Arithmetic and combinatorial properties of generalized Pell sequences

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INFORME SOBRE LA SUSTENTACIÓN

Cumplimiento de Objetivos: Se cumplieron la totalidad de los objetivos propuestos en su proyecto de investigación.

Desarrollo Metodológico: Las principales herramientas utilizadas en esta investigación doctoral son propiedades de las sucesiones lineales recurrentes, cotas inferiores para formas lineales en logaritmos de números algebraicos y una versión del método de reducción de Baker-Davenport proveniente de aproximación diofántica. También se usan propiedades de matrices de Riordan con el fin de deducir identidades y algunos modelos. Con estas herramientas matemáticas se abordan cinco problemas de investigación, extendiendo, en algunos casos, resultados previos que se conocían en la literatura.

Logros del Trabajo o Aportes: Como productos de la investigación doctoral se escribieron cinco artículos, los cuales fueron publicados en revistas internacionales especializadas en el área. Adicionalmente, el estudiante José Luis Herrera presentó resultados de su tesis doctoral en varios eventos, entre los que se destaca el "International Conference on Fibonacci Numbers and their Applications", realizado en Bosnia (online) en julio del 2020.

Se considera el Trabajo de Grado de alto valor académico para que se le confiera:

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A mi madre Luz Dary y a mi padre Victor Manuel,
a mi hermano Ivan Santiago y a mi querida Lorena.

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Resumen

Una generalización de la sucesión de Fibonacci es la sucesión $k-$Fibonacci $F^{(k)}$ cuyos primeros k términos son $0, \ldots, 0, 1$ y cada término de ahí en adelante es la suma de los k términos anteriores. La sucesión $k-$Pell $P^{(k)}$, la cual es una generalización de la clásica sucesión de Pell, se puede definir de manera similar. Aunque la sucesión $F^{(k)}$ ha sido ampliamente estudiada en los últimos años por varios autores, muy poco se sabe sobre $P^{(k)}$. En esta tesis investigamos $P^{(k)}$ y presentamos relaciones de recurrencia, una fórmula tipo Binet y diferentes propiedades aritméticas para la anterior familia de sucesiones. También deducimos modelos combinatorios e identidades que involucran números generalizados tipo Fibonacci y estudiamos algunos problemas diofánticos con las sucesiones $F^{(k)}$ y $P^{(k)}$. Específicamente, encontramos todos los números de Fibonacci generalizados que son números curiosos y caracterizamos $P^{(k)} \cap F^{(\ell)}$ para $k, \ell \geq 2$, extendiendo resultados previos conocidos en algunos casos particulares de k y ℓ. Adicionalmente, determinamos todos los términos de $F^{(k)}$ cercanos a una potencia de 2, generalizando un trabajo previo de Chern y Cui que investigó los números de Fibonacci cercanos a una potencia de 2. Las principales herramientas matemáticas utilizadas en nuestra investigación son la teoría de Baker de formas lineales en logaritmos y una versión del método de reducción de Baker–Davenport perteneciente a la teoría de aproximación Diofántica.

Frases y palabras clave: Número generalizado de Fibonacci, número generalizado de Pell, función generatríz, arreglo Riordan, forma lineal en logaritmos, método de reducción, repdigit.

Abstract

A generalization of the well–known Fibonacci sequence is the $k-$Fibonacci sequence $F^{(k)}$ whose first k terms are $0, \ldots, 0, 1$ and each term afterwards is the sum of the preceding k terms. The $k-$Pell sequence $P^{(k)}$, which is a generalization of the classical Pell sequence, can be defined similarly. Although the sequence $F^{(k)}$ has been extensively studied in recent years by several authors, very little is known about $P^{(k)}$. In this thesis, we investigate $P^{(k)}$ and present recurrence relations, a Binet–type formula and different arithmetic properties for the above family of sequences. Some combinatorial models and interesting identities involving generalized Fibonacci–like numbers are also deduced. We next study some Diophantine problems with the sequences $F^{(k)}$ and $P^{(k)}$. Specifically, we find all curious generalized Fibonacci numbers and characterize $P^{(k)} \cap F^{(\ell)}$ for $k, \ell \geq 2$, extending prior results which dealt with the above problem for some particular cases of k and ℓ. Additionally, we determine all terms of $F^{(k)}$ close to a power of 2, generalizing a previous work of Chern and Cui that investigated the Fibonacci numbers close to a power of 2. The primary mathematical tools used in our investigation are the theory of Baker of linear forms in logarithms and a version of the Baker–Davenport reduction method belonging to the theory of Diophantine approximation.

Keywords: Generalized Fibonacci number, generalized Pell number, generating function, Riordan array, linear form in logarithms, reduction method, repdigit.

Research products

List of publications

The material presented in this thesis covers all the results obtained in the following journal papers:

[25] J. L. Herrera, C. A. Gómez, and J. J. Bravo, *Curious Generalized Fibonacci numbers*, Mathematics **9** (2021), no. 20, 2588.

[26] J. J. Bravo, C. A. Gómez, and J. L. Herrera, *$k-$Fibonacci numbers close to a power of* 2, Quaest. Math. **44** (2021), no. 12, 1681–1690.

[33] J. J. Bravo, J. L. Herrera, and F. Luca, *On a generalization of the Pell sequence*, Math. Bohem. **146** (2021), no. 2, 199–213.

[32] J. J. Bravo, J. L. Herrera, and F. Luca, *Common values of generalized Fibonacci and Pell sequences*, J. Number Theory **226** (2021), 51–71.

[34] J. J. Bravo, J. L. Herrera, and J. L. Ramírez, *Combinatorial interpretation of generalized Pell Numbers*, J. Integer Seq. **23** (2020), no. 2, Article 20.2.1.

Talks

The following talks follow from the results presented in this thesis:

1. *Números de Pell en la sucesión $k-$generalizada de Fibonacci*, VIII Congreso de Álgebra, Teoría de Números, Combinatoria y Aplicaciones (ALTENCOA–8), Popayán Colombia, July 23–27, 2018.

2. *On a generalization of the Pell sequence*, XXII Congreso Colombiano de Matemáticas, Popayán Colombia, June 10–14, 2019.

3. *Una generalización de la sucesión de Pell*, Seminario Sabanero de Combinatoria (SeSaCo), Bogotá Colombia, November 19, 2019.

4. *Dos problemas diofánticos con la sucesión $k-$generalizada de Pell*, VI Seminario Regional de Teoría de Números, Popayán Colombia, March 11–14, 2020.

5. *On a generalization of the Pell sequence*, The Nineteenth International Conference on Fibonacci Numbers and their Applications, Bosnia and Herzegovina (Online), July 21–23, 2020.

Contents

Chapter 1

Introduction

Linear recurrence sequences have become relevant mathematical objects due to their diverse applications in art, science, and technology. In mathematics and computer science, we can find concrete examples in the theory of power series representing rational functions, $k-$regular and automatic sequences, and cellular automata. Within number theory, the solutions of important Diophantine equations form linear recurrence sequences. In this area, the most remarkable sequence is the Fibonacci sequence $(F_n)_{n\geq 0}$ defined by $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$ with initial conditions $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence is one of the most famous and curious numerical sequences in mathematics and has been widely studied in the literature. Another important sequence is the Pell sequence $(P_n)_{n\geq 0}$ which is given by $P_n = 2P_{n-1} + P_{n-2}$ for all $n \geq 2$ with $P_0 = 0$ and $P_1 = 1$. For additional information on linear recurrence sequences, we suggest [6, 57, 87, 88] and the references commented therein.

There is currently an active research field in number theory and combinatorics whose aim is to generalize the Fibonacci and Pell sequences from different points of view. Some generalizations of the Fibonacci sequence preserve the initial conditions and alter the recurrence relation slightly, while others preserve the recurrence relation and alter the initial conditions (see [41, 73, 80, 84, 102, 104, 119]). For example, Milles [102] studied a generalization of the Fibonacci sequence defined by a higher order recurrence relation. Indeed, for an integer $k \geq 2$, he considered the $k-$Fibonacci sequence $F^{(k)} := (F_n^{(k)})_{n\geq 0}$ given by

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for all} \quad n \geq 2,$$

with initial values $F_i^{(k)} = 0$ for $0 \leq i \leq k-2$, and $F_{k-1}^{(k)} = 1$. However, many authors have been working with the shifted sequence $F^{(k)}$ for which the first nonzero value is $F_1^{(k)}$, as seen in [51]. Throughout this thesis we consider the sequence $F^{(k)} := (F_n^{(k)})_{n\geq 2-k}$ defined by the recurrence above but with $F_i^{(k)} = 0$ for $2-k \leq i \leq 0$, and $F_1^{(k)} = 1$ as initial conditions.

There are many papers in the literature dealing with Diophantine equations involving Fibonacci or $k-$Fibonacci numbers. Just to mention a few examples, Gueye et al. [65] showed that $4, 16, 64, 208, 976$ and 1936 are the only $k-$Fibonacci numbers of the form $(3^a \pm 1)(3^b \pm 1)$, where a and b are nonnegative integers. Gueye et al. [66] also found all $k-$Fibonacci numbers which are products of two Fermat numbers, while Hernane et al.[70]) solved the dual problem. In 2021, Gómez et al. [61] formulated the Diophantine equation $(F_n^{(k)})^s = p_1^{a_1} + \cdots + p_t^{a_t}$, where the p_i's form a list of prime numbers in increasing order, and showed that this equation has only finitely many effectively computable solutions. The last result allowed them to extend previos results of Marques [92] and Togbé, Irmak and He [76] that dealt with the Diophantine equation $(F_n)^s = 2^a + 3^b + 5^c$. Finally, Rihane and Togbé [115] determined all $k-$Fibonacci numbers belonging either to the Padovan sequence or the Perrin sequence (for more details about the two last sequences, see [123, A000931 and A001608]). Other interesting properties about $F^{(k)}$ can be found in [89, 134].

Kiliç and D. Taşci [83] considered a generalization of the Pell sequence called the $k-$generalized Pell sequence or, for simplicity, the $k-$Pell sequence $P^{(k)} := (P_n^{(k)})_{n\geq 2-k}$ that satisfies

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)} \quad \text{for all} \quad n \geq 2,$$

with the initial conditions $P_i^{(k)} = 0$ for $2-k \leq i \leq 0$, and $P_1^{(k)} = 1$.

Before proceeding further, we need to mention that the characteristic polynomial of $P^{(k)}$, namely

$$\Phi_k(x) = x^k - 2x^{k-1} - x^{k-2} - \cdots - x - 1,$$

is irreducible over $\mathbb{Q}[x]$ and has just one zero γ outside the unit circle. This single root is located between 2 and 3 (see [135]) and is called the dominant root of $P^{(k)}$.

The $k-$Pell numbers and their properties have been studied (see [33, 80, 81, 83]). Kiliç [81] gave some relations involving Fibonacci and $k-$Pell numbers showing that the $k-$Pell numbers can be expressed as a summation of certain Fibonacci numbers. Kiliç and D. Taşci [83] defined also $P^{(k)}$ in matrix representation and showed that the sums of the $k-$Pell numbers could be derived directly using this representation. Other generalizations of the Pell sequence can be consulted in [50, 129, 133].

In this thesis, we are interested in studying some Diophantine problems involving terms of $F^{(k)}$ and $P^{(k)}$, as well as arithmetic and combinatorial properties of generalized Pell numbers. The main tools used here are Baker's theory of linear forms in logarithms of algebraic numbers and a version of the Baker-Davenport reduction method from Diophantine approximation. Some results of continued fractions and the theory of Riordan arrays that have played an important role in enumerative combinatorics over the last three decades are also used.

This research is organized into eigth chapters and is written in such a way that, after reading Chapter 2, the reader should be able to understand each one of the following chapters separately. The present chapter, as the title suggests, is introductory. Chapter 2 is divided into four sections in which we give the main tools and the preliminary results that will be used in this work. Section 2.1 focuses on an overview of linear recurrence sequences, whereas Section 2.2 collects the main results associated with $k-$Fibonacci numbers. In Section 2.3, we introduce some relevant facts about Diophantine approximation and the theory of continued fractions. This section also includes Lemma 2.7 due to Bravo, Gómez, and Luca [27], which is a small variation of a result showed by Dujella and Petho in [52]. This result will be a key tool when reducing upper bounds of the unknowns of certain Diophantine equations. Finally, Section 2.4 presents a brief survey of linear forms in logarithms of algebraic numbers by emphasizing in a result due to Matveev [96].

In Chapter 3, we investigate the family of sequences $P^{(k)}$ and present recurrence relations, a simplified Binet-style formula and different arithmetic properties. We also deduce interesting identities involving Fibonacci and generalized Pell numbers and show the exponential growth of the $k-$Pell numbers, extending a result known for the case $k = 2$. In Chapter 3 we reproduced [33] and proved the following main result.

Theorem (Chapter 3, Theorem 3.3)**.** *Let $k \geq 2$ be an integer. Then*

(a) *For all $n \geq 2-k$, we have*

$$P_n^{(k)} = \sum_{i=1}^{k} g_k(\gamma_i)\gamma_i^n \quad \textit{and} \quad |P_n^{(k)} - g_k(\gamma)\gamma^n| < 1/2,$$

where $\gamma := \gamma_1, \gamma_2, \ldots, \gamma_k$ are the roots of characteristic polynomial $\Phi_k(x)$ and

$$g_k(z) := \frac{z-1}{(k+1)z^2 - 3kz + k - 1}.$$

(b) *For all $n \geq 1$, we have $\gamma^{n-2} \leq P_n^{(k)} \leq \gamma^{n-1}$.*

It is noteworthy that Theorem 3.3 has become an important tool for those interested in studying generalized Pell sequences, and has already been used by other researchers to address some Diophantine problems (see e.g., [22, 48, 60, 97, 109, 114]).

In combinatorics, the Fibonacci sequence frequently appears in many enumerative models, which provide simple and intuitive proofs of identities involving them. By way of example, F_{n+2} can be interpreted either as the number of subsets of $\{1, 2, \ldots, n\}$ without consecutive integers, or the number of binary strings of length n without consecutive integers. In a similar spirit, F_{n+2} counts the number of sequences of 1's and 2's whose sum is n, and likewise F_{n+2} enumerates the ways to tile a $1 \times n$ rectangle with 1×1 squares and 1×2 dominoes. Heberle [69] extended the above idea and showed that there are $F_{n+1}^{(k)}$ distinct ways to tile a $1 \times n$ rectangle with 1×1 squares and dominoes with length at most k. Rispoli [117] showed also that $k-$Fibonacci numbers count the vertices of a polytope made by the convex hull of the set of $\{0, 1\}$-vectors having d entries and no consecutive k ones. Additional combinatorial interpretations of these numbers are discussed in Chapter 4.

The Pell sequence also has a lot of combinatorial interpretations. To mention one example, the Pell number P_{n+1} counts the tilings of a $1 \times n$ rectangle with dominoes and two colors of squares. A bijective argument allows us to show that P_{n+1} also counts the number of bi–colored compositions of a positive integer n. By a *bi–colored composition* of a positive integer n we mean a sequence of positive integers $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_\ell)$ such that $\sigma_1 + \sigma_2 + \cdots + \sigma_\ell = n$, $\sigma_i \in \{\mathtt{1}, \mathtt{2}\}$, and the summand $\mathtt{1}$ can come in one of two different colors. The reader interested in the long and rich history of compositions can find more information in [72], and for more combinatorial models of Pell numbers see [13, 88].

On the other hand, a *lattice* is an infinite arrangement of points of some Euclidean space with a specific pattern. The simplest example of a lattice is $\mathbb{Z}^n$ which is formed by all points in $\mathbb{R}^n$ with integral coordinates. In the integer context, a *step set* $\mathscr{L}$ is a finite set of vectors of $\mathbb{Z}^n$ which are called *steps*. With this terminology we say that an $n-$*step lattice path* is a sequence of vectors $v = (v_1, v_2, \ldots, v_n)$, such that v_j is in $\mathscr{L}$. Geometrically, it may be interpreted as a sequence of points $w = (w_0, w_1, \ldots, w_n)$, where $w_i \in \mathbb{Z}^2$ (or another starting point), and $w_i - w_{i-1} = v_i$ for $i = 1, 2, \ldots, n$. Lattice paths occur naturally in many areas of mathematics and have applications in physics, computer science, integer programming, cryptanalysis, crystallography, and sphere packing. We refer the reader to [7, 75] for further details on lattice paths.

Chapter 4 deals with combinatorial aspects of generalized Pell sequences and a specific family of lattice paths. This research uses the concept of Riordan arrays (introduced by

Shapiro et al. [120]) which is nowadays a central tool in algebraic combinatorics. In broad terms, Riordan arrays generalize the properties of the Pascal triangle (see [118]), and play an important role inside several proofs of combinatorial identities (see [99, 124]).

In a more formal mathematical language, a *Riordan array* is an infinite lower triangular array $(d_{n,k})_{n,k\in\mathbb{N}}$ defined by two formal power series $(d(t), h(t))$ satisfying the relation

$$d_{n,k} = [t^n]d(t)(h(t))^k.$$

Note that $d(t)(h(t))^k$ is the generating function of the kth column of the Riordan array $(d_{n,k})_{n,k\in\mathbb{N}}$. Additionally, the set of proper Riordan arrays can be equipped with a group structure which allows us to establish an interpretation of the multiplication of two Riordan arrays in terms of formal power series.

Chapter 4 is based on a paper authored together with Bravo and Ramírez [34] where we use Riordan arrays to prove the following results.

Theorem (Chapter 4, Theorem 4.4). *The $k-$Pell number $P_{n+1}^{(k)}$ counts the number of lattice paths from the point $(0,0)$ to $(n-i,i)$ for $i=0,1,\ldots,n$, with step set*

$$S_k = \{H=(1,0),\ V=(0,1),\ D_1=(1,1),\ D_2=(1,2),\ \ldots,\ D_k=(1,k)\}.$$

Theorem (Chapter 4, Theorem 4.5). *The $k-$Pell numbers $P_n^{(k)}$ coincide with the sum of the elements on rising diagonal lines in the Riordan array*

$$\left(\frac{1}{1-2x}, x\frac{1+x+x^2+\cdots+x^{k-2}}{1-2x}\right).$$

Chapter 4 also focuses on *generalized bi–colored compositions* of a positive integer. These compositions extend the idea of classical compositions by letting the summand 1 take two colors. Using this concept, we show the following theorem.

Theorem (Chapter 4, Theorem 4.7). *The generalized Pell number $P_{n+1}^{(k)}$ counts the number of compositions of n with parts in the set $\{1,2,\ldots,k\}$ such that the summand 1 can take two colors.*

As we stated above, Chapters 3 and 4 deal with arithmetical and combinatorial aspects associated with generalized Pell sequences. In Chapter 5, we discuss the intersection between two linear recurrence sequences, which is a classical problem in number theory. On this problem, Mignotte [100] showed that if $\mathcal{U} := (u_n)_{n\geq 0}$ and $\mathcal{V} := (v_m)_{n\geq 0}$ are two linear recurrence sequences, then under some weak technical assumptions, the

Diophantine equation $u_n = v_m$ has only finitely many solutions in positive integers n and m. What he proved is that the intersection of two linear recurrence sequences is finite unless the roots of their characteristic polynomials are multiplicatively dependent. Thus, if the roots of the characteristic polynomials are multiplicatively independent, then Mignotte's result guarantees that $\mathcal{U} \cap \mathcal{V}$ is finite, so the challenge here is to determine what the intersection is.

The intersection between linear recurrence sequences has been discussed by many authors and there is currently a vast literature. For instance, Bravo and Luca [36] and Marques [91] showed independently a conjecture proposed by Noe and Post [106] about coincidences between terms of generalized Fibonacci sequences, while Alekseyev [5] characterized the intersection between Fibonacci and Pell numbers. For our part, we extend in some direction Alekseyev's work by finding all generalized Fibonacci numbers that are Pell numbers, and all Fibonacci numbers which are $k-$Pell numbers (see [24, 30, 71]). In Chapter 5, which is a joint work with Bravo and Luca [32], we solve a more general Diophantine equation extending the previous works in [5, 18, 24, 30, 36, 91]. More precisely, we prove the following.

Theorem (Chapter 5, Theorem 5.1)**.** *The only solutions* (n, k, m, ℓ) *of the Diophantine equation*

$$P_n^{(k)} = F_m^{(\ell)},$$

in positive integers $n, k \geq 2, m, \ell \geq 2$ *are:*

(*i*) *the parametric family of solutions* (n, k, m, ℓ) *with* $\ell = 2$*, namely*

$$(n, k, m, \ell) = (t, k, 2t-1, 2) \quad \textit{for} \quad 1 \leq t \leq k+1;$$

(*ii*) *the sporadic solutions:*

$$\begin{array}{rcllll} 1 & = & P_1^{(k)} = F_1^{(\ell)} & \textit{for all} & k \geq 2 & \textit{and} \quad \ell \geq 3; \\ 1 & = & P_1^{(k)} = F_2^{(\ell)} & \textit{for all} & k, \ell \geq 2; & \\ 2 & = & P_2^{(k)} = F_3^{(\ell)} & \textit{for all} & k \geq 2 & \textit{and} \quad \ell \geq 3; \\ 13 & = & P_4^{(k)} = F_6^{(3)} & \textit{for all} & k \geq 3; & \\ 29 & = & P_5 = F_7^{(4)}. & & & \end{array}$$

An interesting fact about the generalized Fibonacci sequence $F^{(k)}$ is that the first k values after the initial conditions are powers of 2, namely

$$F_n^{(k)} = 2^{n-2} \quad \text{for all} \quad 2 \leq n \leq k+1.$$

Throughout this thesis, each element of the set

$$\bigcup_{k\geq 2}\left\{F_n^{(k)}\ :\ 2\leq n\leq k+1\right\}$$

will be called *a trivial power of* 2. It is known that the only nontrivial power of 2 in the Fibonacci sequence is $F_6 = 8$. One proof of this fact follows from Carmichael's Primitive Divisor theorem [42], which states that for $n > 12$, the nth Fibonacci number F_n has at least one prime factor that is not a factor of any previous Fibonacci number. In 2012, Bravo and Luca [35] generalized this problem by showing that there is no nontrivial power of 2 in $F^{(k)}$ for $k \geq 3$. Shortly after, Gómez and Luca [62] delved deeper into the problem of powers of 2 in the family of generalized Fibonacci sequences and studied the equation $F_n^{(k)} = 2^s F_m^{(k)}$ in positive integers n, m and s. We also mention the work of Petho [110] (see also [44]) about perfect powers in the Pell sequence which allows us to establish that the only power of 2 in $P^{(2)}$ is 2. In the same line, one may wonder about powers of 2 in the family of generalized Pell sequences. The following result, which is an immediate consequence of Theorem 5.1, gives us an answer to this question.

Corollary (Chapter 5, Theorem 5.1)**.** *The only power of* 2 *in* $P^{(k)}$ *is* 2*; namely,*

$$P_2^{(k)} = 2 \qquad \textit{for all} \qquad k \geq 2.$$

In Chapter 6, we consider a Diophantine problem involving the notion of closeness between two numbers introduced by Chern and Cui [43]. We say that an integer n is *close* to a positive integer m if the inequality $|n - m| < \sqrt{m}$ holds. Using this notion, Chern and Cui [43] got a nice result in the context of linear recurrence sequences; i.e., by replacing n y m by members of linear recurrence sequences. Specifically, they found all Fibonacci numbers which are close to a power of 2. In a paper co–authored with Bravo and Gómez [26], we extend the previous work [43] and search for generalized Fibonacci numbers that are close to a power of 2. Chapter 6 reproduces [26] and presents the following main result.

Theorem (Chapter 6, Theorem 6.1)**.** *The Diophantine inequality*

$$|F_n^{(k)} - 2^m| < 2^{m/2},$$

in non-negative integers n, k, m *with* $k \geq 2$ *and* $n \geq 1$*, has two parametric families of solutions* (n, k, m) *with* $n, k \geq 2$ *and* $m \geq 0$*, namely*

(*a*) $(n, k, m) = (t, k, t-2)$ *for* $2 \leq t \leq k+1$*, and*

(*b*) $(n, k, m) = (k+2+t, k, k+t)$ *for* $0 \leq t \leq \max\{x \in \mathbb{Z} : 2 + x < 2^{1+(k-x)/2}\}$.

(c) *In addition, we have the sporadic solution* $(n, k, m) = (12, 3, 9)$.

The study of Diophantine equations involving Fibonacci numbers, their generalizations, and repdigits has attracted the attention of many mathematicians during the last years. A repdigit (short for "repeated digit") is a natural number composed of repeated instances of the same digit in its decimal expansion. It all started in 2000 with the Luca's works [90], who found that 55 and 44 are the largest repdigits in the Fibonacci and Lucas sequences, respectively. Later, Bravo and Luca [37] proved that there are no repdigits with at least two digits in $F^{(k)}$ for $k > 3$, which was somehow extended by Alahmadi et al. [3] who determined all $k-$Fibonacci numbers that are concatenations of two repdigits. We can also mention the work of Trojovský [131] who found all Fibonacci numbers with a prescribed block of digits (i.e., numbers of the form $ab \cdots ba \cdots a$). For more research papers on this topic, we refer the reader to [130] and the references therein.

A *curious number* is a palindromic number whose base–ten representation has the form $a \cdots ab \cdots ba \cdots a$. We know little about curious numbers, which can be seen as blocks of three repdigits with first and third blocks equal. In 2021, Borade and Mayle [16] determined all curious numbers that are perfect squares. Blocks of three repdigits in linear recurrence sequences have been also studied. For instance, Erduvan and Keskin [55, 56] characterized Fibonacci and Lucas numbers which are blocks of three repdigits, respectively. From the above result, it is possible to find all curious numbers in the Fibonacci and Lucas sequences. Chapter 7 is a research collaboration with Bravo and Gómez [25] in which we extend the works [3, 131] by finding all $k-$Fibonacci numbers that are curious numbers.

Theorem (Chapter 7, Theorem 7.1). *The only curious generalized Fibonacci number is*

$$F_{11}^{(5)} = 464.$$

The following is a consequence of Theorem 7.1 and tells us that curious numbers never can be a power of 2.

Corollary (Chapter 7, Corollary 7.1). *There are no powers of* 2 *that are curious numbers.*

This completes the sketch of our thesis.

Chapter 2

Preliminaries

In this chapter, we give a brief overview about the theory of linear recurrence sequences focusing on Fibonacci–like sequences. We also provide basic facts about Diophantine approximation and the theory of continued fractions which will be helpful in this thesis. In addition, we present some results about lower bounds for linear forms in logarithms of algebraic numbers (Baker's method).

2.1 Linear recurrence sequences

Linear recurrence sequences permeate a vast number of areas of mathematics, computer science, and have been a central part of number theory for many years. In this section, we begin by defining a linear recurrence sequence, and summarize results required through our investigation.

To begin, let $k \geq 2$ be an integer number. A *linear recurrence sequence of order k* is a sequence of complex numbers

$$u^{(k)} := (u_n^{(k)})_{n\geq 0}$$

which satisfies a recurrence relation of the form

$$u_{n+k}^{(k)} = a_1 u_{n+k-1}^{(k)} + a_2 u_{n+k-2}^{(k)} + \cdots + a_k u_n^{(k)} \qquad \text{for all} \qquad n \geq 0, \tag{2.1}$$

where $a_1, a_2, \ldots, a_k$ are fixed integers which are usually called *coefficients* of $u^{(k)}$. The values $u_0^{(k)}, u_1^{(k)}, \ldots, u_{k-1}^{(k)}$ must be not all zero and are usually called the *initial conditions* of $u^{(k)}$. For instance, when $k = 2$ we denote $u^{(2)} = u$ and the recurrence (2.1) turns into

$$u_{n+2} = a_1\, u_{n+1} + a_2\, u_n \qquad \text{for} \qquad n \geq 0.$$

For the purposes of this thesis, we next mention two particular cases of u. First, when $(u_0, u_1) = (0, 1)$ and $(a_1, a_2) = (1, 1)$, the sequence u is nothing more than the famous *Fibonacci sequence* $(F_n)_{n\geq 0}$, which appears in [123] as A000045 and satisfies the recurrence

$$F_n = F_{n-1} + F_{n-2} \qquad \text{for all} \qquad n \geq 2,$$

with initial conditions $F_0 = 0$ and $F_1 = 1$. The first few Fibonacci numbers are:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots.$$

Second, for the parameters $(u_0, u_1) = (0, 1)$ and $(a_1, a_2) = (2, 1)$, the sequence u is known as the *Pell sequence* $(P_n)_{n\geq 0}$ (see [123, A000129]) which satisfies the relation

$$P_n = 2P_{n-1} + P_{n-2} \qquad \text{for all} \qquad n \geq 2,$$

with initial conditions $P_0 = 0$ and $P_1 = 1$. The first terms of the Pell sequence are:

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, \ldots.$$

An important concept in the development of our work is the *characteristic polynomial* of $u^{(k)}$ which is defined by using the coefficients of the relation (2.1) as follows

$$\Psi_k(x) := x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_{k-1}x - a_k. \tag{2.2}$$

Under the assumption that

$$\Psi_k(x) = \prod_{i=1}^{t} (x - \alpha_i)^{\sigma_i}, \tag{2.3}$$

where $\alpha_1, \ldots, \alpha_t$ are distinct complex numbers and $\sigma_1, \ldots, \sigma_t$ are positive integer whose sum is k. Then there exist uniquely determined polynomials $q_1(x), \ldots, q_t(x)$ with coefficients in $\mathbb{Q}(\alpha_1, \ldots, \alpha_t)$, $\deg q_i(x) \leq \sigma_i - 1$ for $i = 1, \ldots, t$, such that

$$u_n^{(k)} = \sum_{i=1}^{t} q_i(n)\alpha_i^n \qquad \text{for all} \qquad n \geq 0. \tag{2.4}$$

For the proof of (2.4) we refer to [121, Theorem C.1]. The α_i's involved in (2.3) and (2.4) are called the *roots* of the recurrence. In case that $|\alpha_j| > |\alpha_i|$ for all $i \neq j \in \{1, 2, \ldots, t\}$, we say that α_j is the *dominant root* of $u^{(k)}$ and $q_j(x)$ is the *dominant polynomial* of $u^{(k)}$.

A linear recurrence sequence $u^{(k)}$ is said to be *simple* if all the roots of $\Psi_k(x)$ are simple, i.e., if $\sigma_i = 1$ in (2.3) for $i = 1, \ldots, t$. Suppose that $u^{(k)}$ is simple. Then, we can rewrite the expression (2.4) as

$$u_n^{(k)} = \sum_{i=1}^{k} c_i\, \alpha_i^n \qquad \text{for all} \qquad n \geq 0, \tag{2.5}$$

where $c_1, \ldots, c_k$ are nonzero complex numbers. For the initial conditions

$$(u_0^{(k)}, \ldots, u_{k-1}^{(k)}) = (0, \ldots, 0, 1),$$

the constants $c_1, \ldots, c_k$ in (2.5) were determined by Kalman [78]:

$$c_i = \frac{1}{\Psi_k'(\alpha_i)} \qquad \text{for all} \qquad i = 1, \ldots, k.$$

For the Fibonacci sequence, the expression (2.5) takes the following form

$$F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}} \qquad \text{for all} \qquad n \geq 0, \tag{2.6}$$

where $(\varphi, \overline{\varphi}) := ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$ are the roots of the characteristic equation $x^2 - x - 1 = 0$. The identity (2.6) is traditionally called Binet's Formula of the Fibonacci sequence in honor of the mathematician Jacques Binet (1786–1856). Following this tradition, we say that (2.5) is the *Binet-style expansion* of $u^{(k)}$. An identity similar to (2.6) holds for the Pell sequence, namely

$$P_n = \frac{\gamma^n - \overline{\gamma}^n}{2\sqrt{2}} \qquad \text{for all} \qquad n \geq 0, \tag{2.7}$$

where $(\gamma, \overline{\gamma}) = (1+\sqrt{2}, 1-\sqrt{2})$ are the roots of the characteristic equation $x^2 - 2x - 1 = 0$.

We now consider, for each integer $k \geq 2$, the function $h_k(x)$ defined by

$$h_k(x) = (x-1)\Psi_k(x).$$

Clearly, $h_k'(\alpha_i) = (\alpha_i - 1)\Psi_k'(\alpha_i)$ for all $i = 1, \ldots, k$. So, we obtain from (2.5) that

$$u_n^{(k)} = \sum_{i=1}^{k} \left(\frac{\alpha_i - 1}{h_k'(\alpha_i)} \right) \alpha_i^n \qquad \text{holds for all} \qquad n \geq 0.$$

In 2013, Wu and Zang [135, Lemma 1] provided a sufficient condition on the coefficients of a linear recurrence sequence for the existence of a dominant root. The explicit result is shown below.

Lemma 2.1. *Let $k \geq 2$ be an integer and let $a_1, a_2, \ldots, a_k$ be positive integers with*

$$a_1 \geq a_2 \geq \cdots \geq a_k \geq 1.$$

Then, for the polynomial $\Psi_k(x)$ defined in (2.2) we have:

(a) *$\Psi_k(x)$ has exactly one positive real zero $\alpha(k)$ with $a_1 < \alpha(k) < a_1 + 1$.*

(b) *The remaining $k-1$ zeros of $\Psi_k(x)$ lie within the unit circle in the complex plane.*

(c) *$\Psi_k(x)$ is irreducible over $\mathbb{Q}[x]$.*

Note that Lemma 2.1 (a) leads us to call $\alpha(k)$ the dominant root of $\Psi_k(x)$ whenever its coefficients satisfy that $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$.

We finally present a result of Hubálovská et al. [74, Lemma 2 and 3] which describes the behavior of a linear recurrence sequence with a simple positive dominant root.

Lemma 2.2. *Let $u^{(k)}$ be a linear recurrence whose characteristic polynomial has a simple positive dominant root α. Then*

(a) *The dominant polynomial of $u^{(k)}$ is a positive constant.*

(b) *For any non-negative integers i and r, we have*

$$\lim_{n\to\infty}\left(\frac{u_{n+i}^{(k)}}{\alpha^n}\right)^r = K\alpha^{ir},$$

where K is the constant obtained in (a). In particular, $u_n^{(k)} \sim K\alpha^n$.

2.2 Fibonacci–like sequences

For an integer $k \geq 2$, we consider the family of Fibonacci–like sequences

$$G^{(k)} := (G_n^{(k)})_{n\geq 2-k}$$

defined by the linear recurrence

$$G_n^{(k)} = G_{n-1}^{(k)} + G_{n-2}^{(k)} + \cdots + G_{n-k}^{(k)} \qquad \text{for all} \qquad n \geq 2,$$

and initial conditions $G^{(k)}_{-(k-2)} = G^{(k)}_{-(k-3)} = \cdots = G^{(k)}_{-1} = 0$, $G^{(k)}_0 = a$, and $G^{(k)}_1 = b$, where a and b are fixed integers no both zero. Note that the sequence $G^{(k)}$ is a particular case of the sequence $u^{(k)}$ defined in Section 2.1 by using the parameters

$$a_1 = a_2 = \cdots = a_k = 1,$$

$$u^{(k)}_{-(k-2)} = u^{(k)}_{-(k-3)} = \cdots = u^{(k)}_{-1} = 0,\ u^{(k)}_0 = a \quad \text{and} \quad u^{(k)}_1 = b.$$

In particular, when $a = 0$ and $b = 1$, $G^{(k)} = F^{(k)}$ is known as the $k-$*generalized Fibonacci sequence* or, for simplicity, the $k-$*Fibonacci sequence*. We shall refer to $F^{(k)}_n$ as the n*th* $k-$*Fibonacci number*. These numbers are also called *Fibonacci* $k-$*step numbers* or $k-$*bonacci numbers*. Note that $F^{(k)}$ is a family of sequences where each new choice of k produces a distinct sequence. For example, the usual Fibonacci numbers are obtained with $k = 2$. For small values of k, these sequences are called *Tribonacci* ($k = 3$), *Tetranacci* ($k = 4$), *Pentanacci* ($k = 5$), *Hexanacci* ($k = 6$), *Heptanacci* ($k = 7$) and *Octanacci* ($k = 8$). The $k-$generalized Fibonacci sequences for $k = 3, 4, \ldots, 8$ can be found in [123] as sequences A000073, A000078, A001591, A001592, A001592 and A122189, respectively. The first values of these numbers for $2 \leq k \leq 8$ and $n \geq 1$ are listed in Table 2.1.

Table 2.1: First nonzero $k-$Fibonacci numbers

k	Name	First nonzero terms
2	Fibonacci	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ...
3	Tribonacci	1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, ...
4	Tetranacci	1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, 5536, ...
5	Pentanacci	1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3525, 6930, ...
6	Hexanacci	1, 1, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976, 1936, 3840, 7617, ...
7	Heptanacci	1, 1, 2, 4, 8, 16, 32, 64, 127, 253, 504, 1004, 2000, 3984, 7936, ...
8	Octanacci	1, 1, 2, 4, 8, 16, 32, 64, 128, 255, 509, 1016, 2028, 4048, 8080, ...

An interesting fact about $F^{(k)}$ is that the first $k + 1$ nonzero terms are powers of 2, namely

$$F^{(k)}_n = 2^{\max\{0, n-2\}} \quad \text{for all} \quad 1 \leq n \leq k + 1, \tag{2.8}$$

while the next term is $F^{(k)}_{k+2} = 2^k - 1$. In fact, the inequality

$$F^{(k)}_n < 2^{n-2} \quad \text{holds for all} \quad n \geq k + 2 \tag{2.9}$$

(see [35]). In general, Cooper and Howard [45, Theorem 2.4] proved the following nice formula.

Lemma 2.3. *For $k \geq 2$ and $n \geq k+2$,*

$$F_n^{(k)} = 2^{n-2} + \sum_{j=1}^{\lfloor \frac{n+k}{k+1} \rfloor - 1} C_{n,j}\, 2^{n-(k+1)j-2},$$

where

$$C_{n,j} = (-1)^j \left[\binom{n-jk}{j} - \binom{n-jk-2}{j-2} \right].$$

In the above lemma, we used the convention that $\binom{a}{b} = 0$ if either $a < b$ or if one of a or b is negative and denote $\lfloor x \rfloor$ the greatest integer less than or equal to x. For example, assuming that $k+2 \leq n \leq 2k+2$, Cooper and Howard's formula becomes the identity

$$F_n^{(k)} = 2^{n-2} - (n-k) \cdot 2^{n-k-3} \quad \text{for all} \quad k+2 \leq n \leq 2k+2. \tag{2.10}$$

Another consequence of Lemma (2.3) is the following estimate due to Bravo, Gómez and Luca (see [24, Section 3.3] or [28, Lemma 3]).

Lemma 2.4. *Let $k \geq 2$ and suppose that $r < 2^{k/2}$. Then*

$$F_r^{(k)} = 2^{r-2}(1 + \zeta(r,k)) \quad \textit{where} \quad |\zeta(r,k)| < \frac{1}{2^{k/2}}.$$

On the other hand, Lemma 2.1 implies that the characteristic polynomial of $F^{(k)}$, namely

$$\Psi_k(x) := x^k - x^{k-1} - \cdots - x - 1,$$

is irreducible in $\mathbb{Q}[x]$ and has just one zero real $\alpha := \alpha(k)$ outside the unit circle. The other roots are strictly inside the unit circle, so $\alpha(k)$ is a Pisot number[1] of degree k. Moreover, it is well–known that

$$2(1 - 2^{-k}) < \alpha(k) < 2 \tag{2.11}$$

(see [134, Lemma 3.6]). Thus $\alpha(k)$ approaches 2 as k tends to infinity.

Now, let us consider the function

$$f_k(x) = \frac{x-1}{2+(k+1)(x-2)}, \tag{2.12}$$

[1]A *Pisot number* is a positive algebraic integer greater than 1, all of whose conjugates have absolute value less than 1.

for an integer $k \geq 2$ and $x > 2(1-2^{-k})$. It is easy to see that the inequalities

$$1/2 < f_k(\alpha) < 3/4 \qquad \text{and} \qquad |f_k(\alpha_i)| < 1, \qquad 2 \leq i \leq k \tag{2.13}$$

hold, where $\alpha := \alpha_1, \alpha_2, \ldots, \alpha_k$ are all the zeros of $\Psi_k(x)$. So, by computing norms from $\mathbb{Q}(\alpha)$ to $\mathbb{Q}$, we deduce that the number

$$f_k(\alpha) \text{ is not an algebraic integer.} \tag{2.14}$$

Proofs of (2.13) and (2.14) can be found in [27, Lemma 2 (i)]. Another useful estimation associated with the norm of $f_k(\alpha)$ over $\mathbb{Q}(\alpha)$ is the following (for a proof see [23, 62])

$$N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(f_k(\alpha)) < \frac{2^{k+1}k^k - (k+1)^{k+1}}{k-1} \qquad \text{for all} \qquad k \geq 2. \tag{2.15}$$

With the above notation, Dresden and Du [51] showed that

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha_i)\alpha_i^{n-1} \qquad \text{and} \qquad \left|F_n^{(k)} - f_k(\alpha)\alpha^{n-1}\right| < \frac{1}{2} \tag{2.16}$$

hold for all $n \geq 1$ and $k \geq 2$. This allows us to write

$$F_n^{(k)} = f_k(\alpha)\alpha^{n-1} + e_k(n) \quad \text{where} \quad |e_k(n)| < 1/2, \tag{2.17}$$

for all $n \geq 1$ and $k \geq 2$. When $k = 2$, one can easily prove by mathematical induction that

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \qquad \text{for all} \qquad n \geq 1. \tag{2.18}$$

Bravo and Luca [37] proved that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1} \quad \text{holds for all} \quad n \geq 1 \quad \text{and} \quad k \geq 2, \tag{2.19}$$

which shows that (2.18) holds for the $k-$Fibonacci sequence as well.

In addition, since $G^{(k)}$ and $F^{(k)}$ have the same recurrence relation, one may think that there is some relationship between them. In this sense, Bravo and Luca [39] proved that

$$G_n^{(k)} = aF_{n+1}^{(k)} + (b-a)F_n^{(k)}.$$

On the other hand, if we put $a = 2$ and $b = 1$, then $G^{(k)}$ is known as the $k-$*Lucas sequence* denoted by $L^{(k)} := (L_n^{(k)})_{n \geq 2-k}$. The first few values of nonzero $k-$Lucas numbers are listed in Table 2.2. In the special case where $k = 2$, we get the usual *Lucas sequence* (see [123, A000032]). We point out that analogous properties to those mentioned before hold for the $k-$Lucas numbers (see [39, Lemma 2]). For more details on $k-$Lucas sequences, we refer the reader to [17, 113, 116].

Table 2.2: First nonzero $k-$Lucas numbers

k	Name	First nonzero terms
2	Lucas	2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364,...
3	3−Lucas	2, 1, 3, 6, 10, 19, 35, 64, 118, 217, 399, 734, 1350, 2483, 4567, ...
4	4−Lucas	2, 1, 3, 6, 12, 22, 43, 83, 160, 308, 594, 1145, 2207, 4254, 8200, ...
5	5−Lucas	2, 1, 3, 6, 12, 24, 46, 91, 179, 352, 692, 1360, 2674, 5257, 10335, ...

2.3 Diophantine approximation

In the following chapters, we will study some Diophantine problems by applying lower bounds on linear forms in logarithms which is one of the modern and effective methods for solving Diophantine equations. However, the constants appearing in the lower bounds that the theory provides for linear forms in logarithms are rather large. Therefore, we look for a procedure for reducing bounds to a size that can be more easily handled. In general, the procedure used in this thesis to achieve this goal is the so–called Baker–Davenport reduction method coming from the properties of the continued fractions. In this section, we present a survey of the theory of continued fractions and some basic elements of Diophantine approximation, which are necessary to understand the reduction tools used throughout this work. The main references for this section are [1, Chapter 1, Sections 1.2 and 1.3] and [125, Chapter 1, Section 1].

2.3.1 Good and bad approximations

Diophantine approximation is a branch of number theory which addresses the question of how to approximate a real number through rational numbers. Since the rational numbers are a dense subset of the real numbers, every real number α can be approximated by a sequence of rational numbers converging to α. However, such a sequence for an irrational number has usually denominators that grow very fast. For instance, the sequence $(a_n)_{n\geq 1}$ defined by

$$a_n = \left(1+\frac{1}{n}\right)^n \qquad \text{for all} \qquad n \geq 1,$$

converges to the Euler number e, but its denominators are big, namely

$$(a_n)_{n\geq 1} = \left\{2, \frac{9}{4}, \frac{64}{27}, \frac{625}{256}, \frac{7776}{3125}, \frac{117649}{46656}, \frac{2097152}{823543}, \frac{43046721}{16777216}, \frac{1000000000}{387420489}, \ldots\right\}.$$

Such approximations could improve by requiring rational numbers with comparatively small denominators. One notable contribution in this research line is due to Dirichlet (1805–1859), and his main result is announced below (for a detailed proof, see [1, Theorem 1.1]).

Theorem 2.1. *Let α be a real number and let N be a positive integer. Then, there exists a rational number p/q with $0 < q \leq N$, satisfying the inequality*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{qN} \qquad \left(\leq \frac{1}{q^2}\right).$$

As a consequence of Theorem 2.1, irrational numbers can be distinguish clearly from rational numbers. Namely, the first ones are approximated by infinitely many rational numbers p/q with an error less than $1/q^2$, while the second ones do not satisfies such property. The next corollary of Dirichlet Theorem formalizes the above remark and its proof can be found in [1, Corollary 1.2].

Corollary 2.1. *Let α be a real number.*

(*a*) *If α is an irrational number, then there are infinitely many rational numbers p/q with $q > 0$ such that*
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

(*b*) *For any rational number α and $C > 0$, the inequality*
$$\left|\alpha - \frac{p}{q}\right| < \frac{C}{q^2}$$
is satisfied for only finitely many rational numbers p/q.

Inspired by Corollary 2.1, we say that a rational number p/q with $q > 0$ provides a *good approximation* of the irrational number α if the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2} \tag{2.20}$$

holds. Whereas a real number α is said to be *badly approximable* if there exists a constant $C > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^2} \tag{2.21}$$

holds for all rational number $p/q \neq \alpha$.

We next introduce the concept of continued fractions which play an important role in the process of approximating irrational numbers with rational ones.

2.3.2 Continued fractions

The *Euclidean algorithm* is the oldest and best–known method to compute the greatest common divisor of two integers numbers. To illustrate this procedure, we consider two integer numbers p and q, with $q > 0$, and use the division algorithm recursively

$$\begin{array}{lcll} p & = & a_0\, q + r_0, & 0 < r_0 < q, \\ q & = & a_1\, r_0 + r_1, & 0 < r_1 < r_0, \\ r_0 & = & a_2\, r_1 + r_2, & 0 < r_2 < r_1, \\ & \vdots & & \\ r_{n-2} & = & a_n\, r_{n-1}, & 0 = r_n < r_{n-1} < \cdots < r_0 < q, \end{array} \tag{2.22}$$

where all the a_i's and r_i's are integers. It is well–known that the greatest common divisor of p and q is the last nonzero residue r_{n-1}. Additionally, by rewriting the equalities (2.22) it follows that

$$\begin{aligned} \frac{p}{q} &= a_0 + \frac{r_0}{q}, && 0 < \frac{r_0}{q} < 1, \\ &= a_0 + \frac{1}{q/r_0}, && 1 < \frac{q}{r_0} = a_1 + \frac{r_1}{r_0}, \\ &= a_0 + \frac{1}{a_1 + r_1/r_0}, && 0 < \frac{r_1}{r_0} < 1, \\ &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{r_0/r_1}}, && 1 < \frac{r_0}{r_1} = a_2 + \frac{r_2}{r_1}. \end{aligned}$$

Continuing this way, we arrive at

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}.$$

The right–hand side of the above expression is called the *continued fraction expansion* of the rational number p/q.

In general, a *finite continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}, \tag{2.23}$$

where each a_i is a real number and $a_i > 0$ for $i \geq 1$. To make the writing easier, we also use the abbreviate notation $[a_0, a_1, \ldots, a_n]$. In particular, when each a_i is an integer number, we obtain a *simple continued fraction.* It is worth mentioning that by truncating a continued fraction $[a_0, a_1, \ldots, a_n]$ at the $k-$th place with $0 \leq k \leq n$, we get another continued fraction $C_k := [a_0, a_1, \ldots, a_k]$ called the kth *convergent* of $[a_0, a_1, \ldots, a_n]$.

Evidently, every finite simple continued fraction represents a rational number, and conversely, we have already seen before that every rational number can be written as a finite simple continued fraction. The discussion above allows us to identify rational numbers with finite simple continued fractions, but such an identification is not unique. For instance, for every integer n, there are exactly two different simple continued fraction expansions representing n, namely $n = [n]$ and $n = [n-1, 1]$. Furthermore, a rational r can be represented by either $[a_0, a_1, \ldots, a_n]$ with $a_n \geq 2$, or $[a_0, a_1, \ldots, a_{n-1}, a_n - 1, 1]$. We next illustrate this situation:

$$5 = [5] = [4, 1] \qquad \text{and} \qquad 1/2 = [0, 2] = [0, 1, 1].$$

On the other hand, for a sequence $(a_n)_{n\geq 0}$ of real numbers with $a_i > 0$ for $i \geq 1$, one can define the sequences $\mathfrak{p} := (p_n)_{n\geq -1}$ and $\mathfrak{q} := (q_n)_{n\geq -1}$ by

$$p_n := a_n p_{n-1} + p_{n-2}, \qquad \text{and} \qquad q_n := a_n q_{n-1} + q_{n-2} \qquad \text{for all} \qquad n \geq 1$$

joint with the initial conditions $(p_{-1}, p_0) = (1, a_0)$ and $(q_{-1}, q_0) = (0, 1)$.

Keeping the previous notation and using induction, it is possible to show that for any positive real number ζ,

$$[a_0, a_1, \ldots, a_{n-1}, \zeta] = \frac{\zeta\, p_{n-1} + p_{n-2}}{\zeta\, q_{n-1} + q_{n-2}}.$$

Moreover, the terms of $\mathfrak{p}$ and $\mathfrak{q}$ satisfy the relation

$$\frac{p_k}{q_k} = C_k \qquad \text{for all} \qquad k \geq 0.$$

By Induction, we can also deduce that

$$\prod_{i=1}^{n} \begin{bmatrix} a_i & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_n & q_{n-1} \\ q_n & p_{n-1} \end{bmatrix} \qquad \text{holds for all} \qquad n \geq 0.$$

Thus, the convergents can be easily retrieved by matrix multiplication.

We next want to find a representation for irrational numbers using continued fractions, but first we need to introduce some basic properties.

Lemma 2.5. *Given a sequence of integer numbers* $(a_n)_{n\geq 0}$ *with* $a_i > 0$ *for* $i \geq 1$*, let* $C_n = [a_0, a_1, \ldots, a_n]$*. Then the following hold:*

(a) $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1}$ *for all* $k \geq 0$,

(b) $C_k - C_{k-1} = \dfrac{(-1)^{k+1}}{q_{k-1} q_k}$ *for all* $k \geq 1$,

(c) $p_k q_{k-2} - p_{k-2} q_k = (-1)^k a_k$ *for all* $k \geq 1$,

(d) $C_k - C_{k-2} = \dfrac{(-1)^k a_k}{q_{k-2} q_k}$ *for all* $k \geq 2$,

(e) $q_1 = a_1 \leq 1$, $q_k \leq q_{k-1} + q_{k-2}$, *and* $1 = q_0 \leq q_1 < q_2 < q_3 < \cdots$.

As an immediate consequence of Lemma 2.5 (b), we get that

$$C_0 < C_2 < C_4 < \cdots, \qquad \text{and} \qquad \cdots < C_5 < C_3 < C_1.$$

By Lemma 2.5 (b), we have that $C_{2i} < C_{2i+1}$ for $i \geq 0$, and so $C_{2i} < C_{2j+1}$ for all i, j (it is a good exercise for the reader). The last inequality implies that

$$|C_{2n+1} - C_{2n}| = \frac{1}{q_{2n+1} q_{2n}} \longrightarrow 0,$$

where we have taken into account that $q_n \longrightarrow \infty$ by Lemma 2.5 (e). We are thus deducing that the even and odd convergents are arbitrarily close. That is,

$$\lim_{n\to\infty} C_{2n} = \lim_{n\to\infty} C_{2n+1} = \alpha \qquad \text{exists},$$

and

$$C_{2i} < \alpha < C_{2j+1} \qquad \text{for all} \qquad i, j \geq 0. \tag{2.24}$$

Observations in the previous paragraph allow us to extend the idea of a simple continued fraction to an infinite number of terms. For that, let $(a_n)_{n\geq 0}$ be an integer sequence with $a_i > 0$ for $i \geq 1$. We define the real number $\alpha := [a_0, a_1, a_2, \ldots]$ as

$$\alpha = \lim_{n\to\infty} C_n = \lim_{n\to\infty} [a_0, \ldots, a_n].$$

In this context, C_n is usually called the nth *convergent* of α, and $[a_0, a_1, a_2, \ldots]$ a *simple infinite continued fraction.*

Using the information given above, one can prove that any infinite simple continued fraction is an irrational number (see [1, Theorem 1.15 (1)]). In addition, Leonhard Euler (1707–1783) showed that any irrational number can be expressed as an infinite simple continued fraction. So Euler's result reads as follows (a proof can be found in [1, Theorem 1.15 (3)] or [103, Exercise 8.2.3]).

Lemma 2.6. *Let $x = x_0$ be a positive irrational number. Define the sequence $(a_i)_{i\geq 0}$ recursively as follows:*

$$a_i = \lfloor x_i \rfloor, \quad \text{and} \quad x_{i+1} = \frac{1}{x_i - a_i} \qquad \text{for} \qquad i \geq 0.$$

Then, $x = [a_0, a_1, a_2, \ldots]$ is a representation of x as a continued fraction.

In summary we can conclude that there is a one-to-one correspondence between

- all (finite and infinite) continued fractions $[a_0, a_1, a_2, \ldots]$ with an integer a_0 and positive integers a_i for $i \geq 1$ (and the last term $a_n > 1$ in the case of finite continued fractions), and
- real numbers.

2.3.3 The best approximation

Let us recall that our main objective is to find good approximations for an irrational number through rational numbers. In order to achieve it, we shall show that the convergents of the continued fraction expansion of an irrational number provide the best and unique good approximation in the sense of inequality (2.20).

In the sequel of this section, we assume that $\alpha = [a_0, a_1, a_2, \ldots]$ is an irrational number, and C_n is its nth convergent. By (2.24) we know that α lies between two consecutive convergents, which combined with Lemma 2.5 (b) gives

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} \leq \frac{1}{q_n^2} \qquad \text{for all} \qquad n \geq 0. \tag{2.25}$$

So, C_n provides a good approximation to α. Since the sequence of convergents $(C_n)_{n\geq 0}$ converges to α, the approximation gets better in each step, i.e.,

$$\left|\alpha - \frac{p_n}{q_n}\right| < \left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right| \qquad \text{holds for all} \qquad n \geq 1. \tag{2.26}$$

On the other hand, in Diophantine approximation is well–known that given a rational number p/q with $q > 0$,

$$|q\alpha - p| < |q_n\alpha - p_n| \qquad \text{implies} \qquad q \geq q_{n+1}. \tag{2.27}$$

Property (2.27) tells us that $C_n = p_n/q_n$ is the fraction, among all fractions whose denominator does not exceed q_{n+1}, that provides the best approximation to α. Formally,

$$\left|\alpha - \frac{p}{q}\right| < \left|\alpha - \frac{p_n}{q_n}\right| \qquad \text{implies} \qquad q \geq q_{n+1}. \tag{2.28}$$

We end this section by giving a famous result of Legendre (1752–1833) which is one of the main reasons for studying continued fractions. Legendre's result say that good approximations of irrational numbers by rational numbers are given by the convergents of continued fraction, and these are actually uniques in a certain sense (for more details of Theorem 2.2, and properties (2.25) until (2.28), we refer the reader to [1, Section 1.3]).

Theorem 2.2 (Legendre's Theorem). *Let $\alpha = [a_0, a_1, a_2, \ldots]$ be an irrational number and let p/q be a rational number in lowest term with $q > 0$. Whenever p/q satisfies*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2},$$

then p/q is a convergent of the continued fraction of α, i.e.,

$$\frac{p}{q} = \frac{p_n}{q_n} \qquad \textit{for some} \qquad n \geq 0.$$

Furthermore,

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{(a_{n+1}+2)q_n^2}.$$

2.3.4 Reduction tools

As mentioned before, most of this thesis deals with solving Diophantine problems involving terms of Fibonacci–like sequences. To do this, we first need to get upper bounds of the involved variables in our Diophantine equations. However, these upper bounds are frequently very large, so it is required to reduce them to a small size in which the solutions can be identified by using a computer. A common technique for reducing these bounds is to apply a variation of a result due to Dujella and Pethő [52] based on the Baker–Davenport reduction method [12]. In this thesis, we shall use the following version given by Bravo, Gómez and Luca (see [27, Lemma 1]).

Lemma 2.7. *Let τ be an irrational number, and let A, B, μ be real numbers with $A > 0$ and $B > 1$. Assume that M is a positive integer. Let p/q be a convergent of the continued fraction of τ such that $q > 6M$ and put $\epsilon := \|\mu q\| - M\,\|\tau q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no solution of the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w}$$

in positive integers u, v and w with $u \leq M$ and $w \geq \log(Aq/\epsilon)/\log B$.

The above lemma cannot be applied when μ is an integer linear combination of 1 and τ, since then $\epsilon < 0$. In this case, we use the following nice property of continued fractions (see [20, Lemma 2.8])

Lemma 2.8. *Let p_n/q_n be the nth convergent of the continued fraction $[a_0, a_1, \ldots]$ of the irrational number γ. Let M be a positive integer and put $a_M := \max\{a_i \mid 0 \leq i \leq N+1\}$ where $N \in \mathbb{N}$ is such that $q_N \leq M < q_{N+1}$. If $x, y \in \mathbb{Z}$ with $x > 0$, then*

$$|x\gamma - y| > \frac{1}{(a_M + 2)x} \qquad \text{for all} \qquad x \leq M.$$

We finish this section with the following analytical tool, whose proof can be found in [67, Lemma 7]), and a simple fact concerning the exponential function. We list them as lemmas for further reference.

Lemma 2.9. *If $m \geq 1$ is an integer and x and T are real numbers such that*

$$T > (4m^2)^m \quad \text{and} \quad \frac{x}{(\log x)^m} < T, \qquad \text{then} \qquad x < 2^m T (\log x)^m.$$

Lemma 2.10. *For any nonzero real number x, we have* (*a*) *If* $x > 0$, *then* $0 < x < |e^x - 1|$.

(*b*) *If* $x < 0$ *and* $|e^x - 1| < 1/2$, *then* $|x| < 2\,|e^x - 1|$.

Proof. The first part of the lemma follows immediately by using the fact that $x < e^x - 1$ for all $x \neq 0$. Now, if $x < 0$ and $|e^x - 1| < 1/2$, then we get $1 - e^x < 1/2$ and so $e^{-x} = e^{|x|} < 2$. Thus, $0 < |x| \leq e^{|x|} - 1 = e^{|x|}|e^x - 1| < 2|e^x - 1|$. □

2.4 Linear forms in logarithms

This section is devoted to showing the connection between the good approximation problem of irrational numbers and the theory of linear forms in logarithms. For achieving this, we first present a summary of fundamental facts of algebraic and transcendental numbers. We next study Baker's theory emphasizing the importance within the development of the theory of exponential Diophantine equations. We conclude by presenting a result due to Matveev [96] that gives us a general lower bound for linear forms in logarithms. This section is based on [125, Chapter 1, Sections 2] and [103].

2.4.1 Algebraic and transcendental numbers

We start by recalling that if K is a field containing another field F, then K is said to be an *extension field* (or simply an *extension*) of F, denoted by K/F. It is clear that the multiplication defined in K makes K into a vector space over F. The dimension of the extension is called the *degree* of the extension, and it is abbreviated by $[K : F]$. The extension is called *finite* if $[K : F] < \infty$, and is said to be *infinite* otherwise.

For the aim of this thesis, we are particularly interested in algebraic number fields. From now on, a field $K \subset \mathbb{C}$ is called an *algebraic number field* if it is a finite extension field of $\mathbb{Q}$. Additionally, a complex number α is said to be an *algebraic number* over $\mathbb{Q}$ (or simply an *algebraic number*) if there exists a nonzero polynomial f over $\mathbb{Q}$ such that $f(\alpha) = 0$. Whereas a *transcendental* number is a complex number that is not algebraic. In particular, if α is the root of a monic polynomial with coefficients in $\mathbb{Z}$, we say that α is an *algebraic integer*. Notice that all algebraic integers are algebraic numbers, but the converse is false. It is also possible to prove that the intersection between rational numbers and algebraic integers coincides with integer numbers.

The *minimal polynomial* $f(x)$ of an algebraic number α is the monic polynomial with rational coefficients of smallest degree such that $f(\alpha) = 0$, and the degree of $f(x)$ is usually called the *degree* of α. Furthermore, if $g(x)$ is another polynomial over $\mathbb{Q}$ such that $g(\alpha) = 0$, then $f(x)$ divides $g(x)$. To simplify the exposition, let us write $\min_{\mathbb{Q}}(\alpha)$ and $\deg(\alpha)$ to indicate the minimal polynomial and the degree of α, respectively. Finally, algebraic numbers over $\mathbb{Q}$ with same minimal polynomial are named *conjugates* over $\mathbb{Q}$.

As has already been mentioned, the problem of how well a real number α can be approximated by rational numbers was solved by means of continued fractions (see Subsection 2.3.3). Now, the main problem is to find sharp lower bounds for the approximations in terms of the denominators used in the estimation. An important contribution to the subject was made by Liouville in 1853, since he showed that algebraic numbers cannot be too well approximated by rationals. For details see [103, Theorem 3.2.1].

Theorem 2.3 (Liouville's theorem)**.** *Let α be a real algebraic number of degree $d \geq 2$. Then there exists a constant $c(\alpha) > 0$, depending only on α, such that*

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c(\alpha)}{q^d}$$

for all rational number p/q.

Theorem 2.3 is transcendental for mathematical culture, since this allowed to elaborate the first explicit transcendental numbers. Years later, in 1873, Hermite proved that e is transcendental, and Ferdinand von Lindemann in 1882 proved the transcendence of π. Moreover, Hermite showed that e^a is transcendental when a is algebraic and nonzero. This property has been generalized to the nowadays called Lindemann-Weierstrass theorem.

Theorem 2.4 (Lindemann-Weierstrass theorem)**.** *Let $\beta_1, \ldots, \beta_n$ be nonzero algebraic numbers and $\alpha_1, \ldots, \alpha_n$ be distinct algebraic numbers. Then*

$$\beta_1 e^{\alpha_1} + \cdots + \beta_n e^{\alpha_n} \neq 0.$$

In particular, from the Lindemann-Weierstrass theorem we obtain that if α is a nonzero algebraic integer, then e^{α}, $\sin\alpha$, and $\cos\alpha$ are transcendental numbers.

2.4.2 Baker's theory

In 1900, at the International Congress of Mathematicians in Paris, Hilbert (1862-1943) presented ten of twenty–three unsolved problems which were influential for the 20th–

century mathematics. According to Hilbert, it would need new machinery and methods for solving these problems.

The seventh problem, entitled "*irrationality and transcendence of certain numbers*", consisted into proving the transcendence of the number α^β for an algebraic $\alpha \neq 0, 1$ and an irrational algebraic β. Hilbert expected that the seventh problem would be solved later than Riemann hypothesis, and Fermat's last theorem. However, this problem was solved independently by Gelfond (1906–1968), and Schneider (1911–1988) in 1935. Before presenting their result, we recall that the (real or complex) numbers $\alpha_1, \ldots, \alpha_n$ are called *linearly dependent* over $\mathbb{Q}$ (equivalently over $\mathbb{Z}$) if there are rational numbers (integer numbers) $r_1, \ldots, r_n$, not all zero, such that

$$r_1\alpha_1 + r_2\alpha_2 + \cdots + r_n\alpha_n = 0.$$

Moreover, if $\alpha_1, \ldots, \alpha_n$ are not linearly dependent over $\mathbb{Q}$ (over $\mathbb{Z}$), they are *linearly independent* over $\mathbb{Q}$ (over $\mathbb{Z}$).

We next formulate the Gelfond–Schneider theorem.

Theorem 2.5. *If $\alpha_1, \alpha_2 \neq 0$ are algebraic numbers such that $\log\alpha_1, \log\alpha_2$ are linearly independent over $\mathbb{Q}$, then $\log\alpha_1$ and $\log\alpha_2$ are linearly independent over the set of algebraic numbers, that is*

$$\beta_1 \log\alpha_1 + \beta_2 \log\alpha_2 \neq 0,$$

for all algebraic numbers β_1, β_2.

It is widely known that the Gelfond–Schneider theorem is equivalent to the following result.

Theorem 2.6. *(Seventh Hilbert's problem) If α and β are algebraic numbers with $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$, then α^β is transcendental.*

On the other hand, a *linear form in logarithms* of algebraic numbers is an expression of the form

$$\beta_0 + \beta_1 \log\alpha_1 + \beta_2 \log\alpha_2 + \cdots + \beta_n \log\alpha_n,$$

where each α_i and β_i are complex algebraic numbers, and log denotes any determination of the logarithm. Note that linear forms in logarithms appear explicitly in Theorem 2.5, and implicitly in Theorem 2.4. In the sequel, we are interested in the degenerate case, which happens when $\beta_0 = 0$ and $\beta_i \in \mathbb{Z}$ for $i \geq 1$. We write usually $\beta_i = b_i$ for $i \geq 1$.

The most prolific research on linear forms in logarithms is due to Baker (1939–2018) (see [95]). He gave an effective lower bound on the absolute value of a nonzero linear form in logarithms of algebraic numbers (for more information, see [8, 9, 10, 11]). This work has allowed to efficiently solve equations whose unknowns are in the exponents. These type of equations are known as *exponential Diophantine equations*. Baker also extended in 1966 the Gelfond-Schneider theorem to arbitrarily many logarithms, which turned into a new powerful analytic tool for solving several open problems in number theory. For his contribution in transcendental number theory and Diophantine geometry, Baker was awarded the Fields Medal in 1970. Finally, for this thesis, it is important to highlight the following theorem since it was the start point for a new branch in number theory called *Baker's theory*. A proof can be found for instance in [8].

Theorem 2.7 (Baker's theorem)**.** *If* $\alpha_1, \ldots, \alpha_n \neq 0, 1$ *are algebraic numbers such that* $\log \alpha_1, \log \alpha_2, \ldots, \log \alpha_n$ *and* $2\pi i$ *are linearly independent over* $\mathbb{Q}$*, then*

$$\beta_0 + \beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 + \cdots + \beta_n \log \alpha_n \neq 0$$

for any algebraic numbers $\beta_0, \beta_1, \ldots, \beta_n$ *that are not all zero.*

We finally comment that Baker's works have allowed to solve a wide variety of Diophantine equations of exponential type. Tijdeman [127, Section 5] and the references therein describes Baker's method in a nutshell, and summaries the main equations worked by him and other authors. We next give some Diophantine equations worked at the moment with Baker's techniques.

- $f(x) = m$, where $f \in \mathbb{Z}[x, y]$ is an irreducible homogeneous polynomial of degree $n \geq 3$ and m is a nonzero integer,
- $y^2 = x^3 + m$, where m is a nonzero integer,
- $f(x) = y^2$, where $f(x) \in \mathbb{Z}[x]$ has at least three simple zeros,
- $f(x) = y^m$, where $f(x) \in \mathbb{Z}[x]$ has at least two simple zeros and $m \geq 3$,
- $1^k + 2^k + \cdots + x^k = y^n$ for a given integer k and unknowns n, x, y,
- $\prod_{i=1}^{k}(x + id) = y^n$ for a given d, k and unknowns n, x, y,
- $(x^m - 1)/(x - 1) = y^n$ in integers x, y, m, n subject to some restriction,
- $u_n = y^m$, where $(u_n)_{n\geq 0}$ is a binary recurrence sequence and m, n, y are unknowns.

2.4.3 Matveev's theorem

We begin by underscoring that Theorem 2.7 tells us that any linear form in logarithms

$$\beta_0 + \beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 + \cdots + \beta_n \log \alpha_n,$$

where each α_i and β_i are nonzero algebraic numbers, vanishes only in trivial cases. Nevertheless, for Diophantine applications, it is not enough to know when a linear form is nonzero. Actually, we need strong enough lower bounds for the absolute value of them. In this direction, Baker in [8] provides an interesting estimation when $\beta_0 = 0$ and $\beta_1, \ldots, \beta_n$ are rational integers, which we present below.

Theorem 2.8. *Let $\alpha_1, \ldots, \alpha_n \neq 0, 1$ be algebraic numbers, and let $b_1, \ldots, b_n$ be rational integers such that*

$$b_1 \log \alpha_1 + b_2 \log \alpha_2 + \cdots + b_n \log \alpha_n \neq 0.$$

Then,

$$|b_1 \log \alpha_1 + b_2 \log \alpha_2 + \cdots + b_n \log \alpha_n| \geq (eB)^{-C},$$

where $B := \max\{|b_1|, \ldots, |b_n|\}$, and C is an effectively computable constant[2] *depending only on n and $\alpha_1, \ldots, \alpha_n$.*

Theorem 2.8 yields the following corollary which has a more convenient presentation for applications.

Corollary 2.2. *Let $\alpha_1, \ldots, \alpha_n \neq 0, 1$ be algebraic numbers, and let $b_1, \ldots, b_n$ be rational integers such that*

$$\alpha_1^{b_1} \alpha_2^{b_2} \cdots \alpha_n^{b_n} - 1 \neq 0.$$

Then,

$$\left|\alpha_1^{b_1} \alpha_2^{b_2} \cdots \alpha_n^{b_n} - 1\right| \geq (eB)^{-C'},$$

where $B := \max\{|b_1|, \ldots, |b_n|\}$, and C' is an effectively computable constant depending only on n and $\alpha_1, \ldots, \alpha_n$.

In the case that $\alpha_1, \ldots, \alpha_n$ are rational numbers, Matveev [96] showed that the constant C' in Corollary 2.2 is equal to

$$\frac{1}{2} e \cdot 30^{n+3} \cdot n^{4.5} \prod_{i=1}^{n} \max\{1, \log \mathrm{H}(\alpha_i)\},$$

[2]The statement "*C is an effectively computable constant*" means that by going through the proof one can compute an explicit value of C.

where the *height* H of a rational number $\alpha = p/q$ with $p, q \in \mathbb{Z}$ and $gcd(p,q) = 1$ is defined by $\mathrm{H}(\alpha) \coloneqq \max\{|p|, |q|\}$.

In order to prove our main results, we need to use a Baker type lower bound for a nonzero linear form in logarithms of algebraic numbers. Such a bound was given by Matveev in [96] and plays an important role in this thesis. Before presenting such a theorem, we recall some basic notions from algebraic number theory.

Let η be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial over the integers

$$a_0 \prod_{i=1}^{d} (x - \eta^{(i)}) \in \mathbb{Z}[x],$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. The *logarithmic height* of η is given by

$$h(\eta) \coloneqq \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max\{|\eta^{(i)}|, 1\}\right).$$

As a first illustration of the logarithmic height function, if $\eta = p/q$ is a rational number with $\gcd(p,q) = 1$ and $q > 0$, then it is not difficult to check that $h(\eta) = \log \max\{|p|, q\}$. Now, let $\eta = \alpha$ be the dominant root of the Fibonacci–like sequence $G^{(k)}$ and let us consider the function f_k defined by (2.12). Knowing that the minimal primitive polynomial of α is $\Psi_k(x)$, that $\mathbb{Q}(\alpha)$ coincides with $\mathbb{Q}(f_k(\alpha))$ and that $|f_k(\alpha^{(i)})| \leq 1$ for all $i = 1, \ldots, k$ and $k \geq 2$, one can prove that

$$h(\alpha) = (\log \alpha)/k < (\log 2)/k \quad \text{and} \quad h(f_k(\alpha)) < 2 \log k \quad \text{for all} \quad k \geq 2. \tag{2.29}$$

See [27] for further details of the proof of (2.29).

The following are some of the properties [132, Property 3.3] of the logarithmic height function $h(\cdot)$, which will be used in the remaining of this document without reference. For η, γ algebraic numbers and $s \in \mathbb{Z}$, we have

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \qquad h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \\ h(\eta) &= h(\eta^{(i)}), \qquad h(\eta^s) = |s| h(\eta) \qquad (s \in \mathbb{Z}). \end{aligned}$$

Our main tool is the following lower bound for a nonzero linear form in logarithms of algebraic numbers due to Matveev [96].

Theorem 2.9 (Matveev's theorem)**.** *Let $\eta_1, \ldots, \eta_t$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, and let $b_1, \ldots, b_t$ be rational integers. Assume that*

$$\Lambda \coloneqq \eta_1^{b_1} \cdots \eta_t^{b_t} - 1 \neq 0.$$

Then

$$|\Lambda| > \exp\left(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1+\log D)(1+\log B)A_1 \cdots A_t\right),$$

where $A_1, \ldots, A_t$ *and* B *are real numbers such that*

$$B \geq \max\{|b_1|, \ldots, |b_t|\}$$

and

$$A_i \geq \max\{Dh(\eta_i), |\log \eta_i|, 0.16\} \qquad \textit{for} \qquad i = 1, \ldots, t.$$

To conclude this section, we give an important estimate that will be used in the following chapters in several applications of Matveev's theorem.

Lemma 2.11. *Let* $k \geq 2$ *and* $s \neq 0$ *be integers and suppose that* $|s| \leq 10^{\varepsilon}$ *for some integer* $\varepsilon \geq 1$*. Then*

$$h\left(9 f_k(\alpha) s^{-1}\right) < \varepsilon \log 10 + 2 \log k.$$

Furthermore, if $\varepsilon = 1$*, then*

$$h\left(9 f_k(\alpha) s^{-1}\right) < 6 \log k.$$

Chapter 3

On a generalization of the Pell sequence

The Pell sequence $(P_n)_{n\geq 0}$ is the second order linear recurrence $P_n = 2P_{n-1} + P_{n-2}$ with initial conditions $P_0 = 0$ and $P_1 = 1$. In this chapter, we investigate a generalization of the Pell sequence called the $k-$generalized Pell sequence which is generated by a recurrence relation of a higher order. We present recurrence relations, the generalized Binet formula and different arithmetic properties for the above family of sequences. Some interesting identities involving the Fibonacci and generalized Pell numbers are also deduced.

3.1 Introduction

The Fibonacci sequence $(F_n)_{n\geq 0}$ is one of the most known and studied sequences in the history of mathematics. Nowadays, there is a wide bibliography dealing with its properties and connections with other areas of knowledge (see e.g. [64, 87, 13]). All this academic effort has led mathematicians to generalize the Fibonacci sequence in many ways; some of them preserve the initial conditions and others preserve the recurrence relation (see [41, 73, 80, 84, 104, 119, 134]).

Among the many generalizations of the Fibonacci sequence, we point out that $F^{(k)}$ is one of the most recently studied sequences. According to Kunth [85], this sequence was studied for the first time by Schlegel [El Progreso Matemático **4** (1894), 173–174]. However, Kessler and Schiff [79] state that the paper of Miles [102] seems to be the oldest well–known formal paper on the subject. Dresden and Du [51] recently found a Binet–style formula for $F^{(k)}$ that can be used to produce the $k-$generalized Fibonacci numbers and interesting arithmetic properties. For more details about $F^{(k)}$, we refer the interested reader to [4, 66, 70] and the references therein.

The Pell $(P_n)_{n\geq 0}$ and the Pell–Lucas $(Q_n)_{n\geq 0}$ sequences are infinite sequences of integers known since ancient times and named in honor of mathematicians Jhon Pell (1611–1685) and François Edouard Anatole Lucas (1842–1891). These numbers may be calculated by means of recurrence relations similar to that for the Fibonacci numbers. The first one is defined by the recurrence

$$P_n = 2P_{n-1} + P_{n-2} \qquad \text{for all} \qquad n \geq 2,$$

with $P_0 = 0$ and $P_1 = 1$ while the second one satisfies

$$Q_n = 2Q_{n-1} + Q_{n-2} \qquad \text{for all} \qquad n \geq 2,$$

with $Q_0 = 2$ and $Q_1 = 2$. The last sequence coincides with entry A002203 in the OEIS [123].

Within the great variety of properties of these sequences, we emphasize that (P_n, Q_n) constitute the solutions of Pell's equation $x^2 - 2y^2 = (-1)^n$ (see [88, Section 2.5]). Furthermore, like Fibonacci and Lucas numbers, Pell and Pell–Lucas sequences are mathematical twins and for this reason, they share similar properties. For example, both sequences grow exponentially and proportionally to powers of the silver ratio $1 + \sqrt{2}$. For further information about Pell and Pell–Lucas numbers, see [15, 82, 87, 88].

In this chapter we study, for an integer $k \geq 2$, a generalization of the Pell sequence which is defined by a recurrence relation of higher order. Here, we consider the *$k-$generalized Pell sequence* or, for simplicity, the *$k-$Pell sequence* $P^{(k)} := (P_n^{(k)})_{n\geq -(k-2)}$ given by the recurrence

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)} \quad \text{for all} \quad n \geq 2, \tag{3.1}$$

with the initial conditions $P_{-(k-2)}^{(k)} = P_{-(k-1)}^{(k)} = \cdots = P_0^{(k)} = 0$ and $P_1^{(k)} = 1$. We should mention that this generalization of the Pell sequence is part of a family of sequences proposed by Kiliç and Taşci [83]. Other generalizations are also known, see e.g. [80, 86, 136, 129].

We shall refer to $P_n^{(k)}$ as the *nth $k-$Pell number.* Note that each new choice of k produces a distinct sequence. For example, when $k = 2$ we obtain the usual Pell sequence $(P_n)_{n\geq 0}$ while $k = 3$ leads to the *Tripell sequence* (see [123, A077939]). In Table 3.1 we present the first nonzero values of the family of sequences $P^{(k)}$.

Table 3.1: First nonzero $k-$Pell numbers

k	Name	First nonzero terms
2	Pell	1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, ...
3	Tripell	1, 2, 5, 13, 33, 84, 214, 545, 1388, 3535, 9003, 22929, 58396, ...
4	4−Pell	1, 2, 5, 13, 34, 88, 228, 591, 1532, 3971, 10293, 26680, 69156, ...
5	5−Pell	1, 2, 5, 13, 34, 89, 232, 605, 1578, 4116, 10736, 28003, 73041, ...
6	6−Pell	1, 2, 5, 13, 34, 89, 233, 609, 1592, 4162, 10881, 28447, 74371, ...

Tripell numbers and generalized Pell numbers have been studied by some authors in [21, 22] and [30, 31, 48, 81, 60, 83, 114], respectively. In 2013, Kiliç [81] gave some relations involving the usual Fibonacci and $k-$Pell numbers showing that some $k-$Pell numbers can be expressed as the summation of the usual Fibonacci numbers. More precisely, if $k+2 \leq n \leq 2k+2$, then

$$P_n^{(k)} = F_{2n-1} - \sum_{j=1}^{n-k-1} F_{2j-1}F_{2(n-k-j)}. \tag{3.2}$$

One the facts that Kiliç [81] used to prove identity (3.2) was that the first $k+1$ nonzero terms in $P^{(k)}$ are Fibonacci numbers with odd index, namely

$$P_n^{(k)} = F_{2n-1} \quad \text{for all} \quad 1 \leq n \leq k+1, \tag{3.3}$$

while the next term is $P_{k+2}^{(k)} = F_{2k+3} - F_1F_2$. The authors of [83] also defined $P^{(k)}$ in matrix representation and showed that the sums of the $k-$generalized Pell numbers could be derived directly using this representation.

Throughout this chapter, we investigate the $k-$generalized Pell sequences and present recurrence relations, a simplified Binet–style formula and different arithmetic properties for $P^{(k)}$. Some interesting identities involving the Fibonacci and generalized Pell numbers are also deduced and some well–known properties of $P^{(2)}$ are generalized to the sequence $P^{(k)}$. We also exhibit a good approximation to the nth $k-$Pell number and show the exponential growth of $P^{(k)}$.

3.2 Preliminary results

First of all, we denote the characteristic polynomial of the $k-$Pell sequence $P^{(k)}$ by

$$\Phi_k(x) = x^k - 2x^{k-1} - x^{k-2} - \cdots - x - 1.$$

In 2013, Wu and Zang [135] showed that if $a_1, a_2, \ldots, a_m$ are positive integers satisfying $a_1 \geq a_2 \geq \cdots \geq a_m$, then for the polynomial

$$p(x) = x^m - a_1 x^{m-1} - a_2 x^{m-2} - \cdots - a_{m-1}x - a_m,$$

we have:

(*i*) The polynomial $p(x)$ has exactly one positive real zero α with $a_1 < \alpha < a_1 + 1$;

(*ii*) The others $m-1$ zeros of $p(x)$ lie within the unit circle in the complex plane.

From the above we deduce that $\Phi_k(x)$ has just one real zero located between 2 and 3. Throughout this document, $\gamma := \gamma(k)$ denotes that single zero which is a Pisot number of degree k since the other zeros of $\Phi(x)$ are strictly inside the unit circle. This important property of γ leads us to call it *the dominant root* of $P^{(k)}$. Since γ is a Pisot number of minimal polynomial $\Phi_k(x)$, it follows that this polynomial is irreducible over $\mathbb{Q}[x]$. To simplify notation, we shall omit the dependence on k of γ whenever no confusion may arise.

We now consider, for each integer $k \geq 2$, the polynomial function $h_k(x)$ defined by

$$\begin{aligned} h_k(x) &= (x-1)\Phi_k(x) \\ &= x^{k+1} - 3x^k + x^{k-1} + 1. \end{aligned} \tag{3.4}$$

Since $P^{(k)}$ is a linear recurrence of order k with characteristic polynomial $\Phi_k(x)$ and $\Phi_k(x)$ divides $h_k(x)$, we deduce that $P^{(k)}$ is also a linear recurrence of order $k+1$ with characteristic polynomial $h_k(x)$. Hence, we obtain our first preliminary result which is a "shift formula" that will be used in what follows.

Theorem 3.1. *Let $k \geq 2$ be integer. Then*

$$P_n^{(k)} = 3P_{n-1}^{(k)} - P_{n-2}^{(k)} - P_{n-k-1}^{(k)} \quad \textit{for all} \quad n \geq 3.$$

As an application of Theorem 3.1, we give alternative proofs of identities (3.3) and (3.2), which have already been proved by Kiliç in [81] as mentioned before.

Let us begin by proving (3.3). We first observe that $P_1^{(k)} = 1 = F_1$ and $P_2^{(k)} = 2 = F_3$; therefore (3.3) is valid for $n = 1, 2$. Let $3 \leq s \leq k$ be an integer and suppose that

$P_\ell^{(k)} = F_{2\ell-1}$ for all $3 \le \ell \le s$. Hence, to complete the proof of (3.3) by mathematical induction, we have to show that $P_{s+1}^{(k)} = F_{2s+1}$.

Since $s+1 \ge 4$, by Theorem 3.1, we have

$$P_{s+1}^{(k)} = 3P_s^{(k)} - P_{s-1}^{(k)} - P_{s-k}^{(k)}. \tag{3.5}$$

By the induction hypothesis $P_s^{(k)} = F_{2s-1}$ and $P_{s-1}^{(k)} = F_{2s-3}$, and taking into account that $P_{s-k}^{(k)} = 0$ because $3-k \le s-k \le 0$, we get

$$P_{s+1}^{(k)} = 3F_{2s-1} - F_{2s-3}. \tag{3.6}$$

We now observe that the recurrence relation of the Fibonacci sequence implies the recursive formula

$$3F_n - F_{n-2} = F_{n+2}, \tag{3.7}$$

which holds for all $n \ge 2$. Consequently, it follows from (3.6) and (3.7) that

$$P_{s+1}^{(k)} = F_{2s+1},$$

as we wanted.

We next prove (3.2). It follows from Theorem 3.1, (3.3) and (3.7) that

$$P_{k+2}^{(k)} = 3P_{k+1}^{(k)} - P_k^{(k)} - P_1^{(k)} = F_{2k+3} - F_1$$

and

$$P_{k+3}^{(k)} = 3P_{k+2}^{(k)} - P_{k+1}^{(k)} - P_2^{(k)} = F_{2k+5} - (F_1F_4 + F_3F_2).$$

Hence, (3.2) is valid for $n = k+2$ and $n = k+3$. Now, let $k+4 \le s \le 2k+1$ be an integer and suppose that

$$P_\ell^{(k)} = F_{2\ell-1} - \sum_{j=1}^{\ell-k-1} F_{2j-1}F_{2(\ell-k-j)} \quad \text{holds for all} \quad k+4 \le \ell \le s.$$

To complete the proof of (3.2) by mathematical induction, we have to prove that

$$P_{s+1}^{(k)} = F_{2s+1} - \sum_{j=1}^{s-k} F_{2j-1}F_{2(s-k-j+1)}. \tag{3.8}$$

Indeed, since $s+1 \ge 7$ we have by Theorem 3.1 that (3.5) holds again. Also, by (3.3) we get that $P_{s-k}^{(k)} = F_{2(s-k)-1}$ because $s-k \in [4, k+1]$. From the above and using the facts that

$$P_s^{(k)} = F_{2s-1} - \sum_{j=1}^{s-k-1} F_{2j-1}F_{2(s-k-j)}$$

and

$$P_{s-1}^{(k)} = F_{2s-3} - \sum_{j=1}^{s-k-2} F_{2j-1}F_{2(s-k-j-1)},$$

we obtain

$$P_{s+1}^{(k)} = (3F_{2s-1} - F_{2s-3}) - \sum_{j=1}^{s-k-2} F_{2j-1}(3F_{2(s-k-j)} - F_{2(s-k-j-1)}) - 3F_{2(s-k-1)-1} - F_{2(s-k)-1}.$$

Consequently, from (3.7) and the last equality we get (3.8).

Another application of Theorem 3.1 will enable us to derive an extended version of (3.2) in the following form:

Theorem 3.2. *Let $k \geq 2$ be an integer. Then*

$$P_n^{(k)} = F_{2n-1} - \sum_{j=1}^{n-k-1} F_{2j}P_{n-k-j}^{(k)},$$

for all $n \geq k+2$.

Note that Theorem 3.2 immediately shows that the *nth k−Pell number* does not exceed the Fibonacci number with index $2n-1$, i.e.,

$$P_n^{(k)} < F_{2n-1} \quad \text{holds for all} \quad k \geq 2 \quad \text{and} \quad n \geq k+2.$$

Proof. We shall prove Theorem 3.2 by induction on n. According to (3.2), we have

$$P_{k+2}^{(k)} = F_{2k+3} - 1 = F_{2k+3} - F_2P_1^{(k)},$$

and

$$P_{k+3}^{(k)} = F_{2k+5} - 5 = F_{2k+5} - F_2P_2^{(k)} - F_4P_1^{(k)}.$$

Then the result holds for $n = k+2$ and $n = k+3$. Let $s \geq k+4$ be an integer and suppose that

$$P_\ell^{(k)} = F_{2\ell-1} - \sum_{j=1}^{\ell-k-1} F_{2j}P_{\ell-k-j}^{(k)} \quad \text{holds for all} \quad k+4 \leq \ell \leq s.$$

We have to prove that

$$P_{s+1}^{(k)} = F_{2s+1} - \sum_{j=1}^{s-k} F_{2j}P_{s+1-k-j}^{(k)}. \tag{3.9}$$

Indeed, by Theorem 3.1 and the induction hypothesis,

$$P_{s+1}^{(k)} = 3F_{2s-1} - 3\sum_{j=1}^{s-k-1} F_{2j}P_{s-k-j}^{(k)} - F_{2s-3} + \sum_{j=1}^{s-k-2} F_{2j}P_{s-1-k-j}^{(k)} - P_{s-k}^{(k)}$$

leading to

$$P_{s+1}^{(k)} = 3F_{2s-1} - F_{2s-3} - 3\sum_{j=2}^{s-k-1} F_{2j}P_{s-k-j}^{(k)} + \sum_{j=1}^{s-k-2} F_{2j}P_{s-1-k-j}^{(k)} - 3F_2P_{s-k-1}^{(k)} - P_{s-k}^{(k)}.$$

From the above we have, after some elementary algebra, that

$$P_{s+1}^{(k)} \;=\; F_{2s+1} - \sum_{j=1}^{s-k-2} \left(3F_{2j+2} - F_{2j}\right) P_{s-k-(j+1)}^{(k)} - 3P_{s-k-1}^{(k)} - P_{s-k}^{(k)},$$

and therefore

$$P_{s+1}^{(k)} \;= F_{2s+1} - \sum_{j=3}^{s-k} F_{2j}P_{s+1-k-j}^{(k)} - F_4P_{s-k-1}^{(k)} - F_2P_{s-k}^{(k)},$$

of which it follows (3.9) as we wanted. □

3.3 Main results

This section is devoted to stating and proving the results concerned with a simplified generalized Binet–style formula for $P^{(k)}$ and its exponential growth in which we prove that the $k-$Pell numbers grow at an exponential rate equal to the dominant root γ, extending a result known for the usual Pell numbers. We also show that a good approximation to the nth $k-$Pell number is just the term of the Binet–style formula involving the dominant root.

We summarize the main results in the following theorem.

Theorem 3.3. *Let $k \geq 2$ be an integer. Then*

(*a*) *For all $n \geq 2-k$, we have*

$$P_n^{(k)} = \sum_{i=1}^{k} g_k(\gamma_i)\gamma_i^n \quad and \quad |P_n^{(k)} - g_k(\gamma)\gamma^n| < 1/2,$$

where $\gamma := \gamma_1, \gamma_2, \ldots, \gamma_k$ *are the roots of characteristic polynomial* $\Phi_k(x)$ *and*

$$g_k(z) := \frac{z-1}{(k+1)z^2 - 3kz + k - 1}. \tag{3.10}$$

(*b*) *For all* $n \geq 1$, *we have*

$$\gamma^{n-2} \leq P_n^{(k)} \leq \gamma^{n-1}. \tag{3.11}$$

In order to prove Theorem 3.3, we establish some lemmas which give us interesting properties of the dominant root of $P^{(k)}$, and we believe are of independent interest.

3.3.1 Generalized Binet–Style formula

In 1982, Kalman [78] proved that if $(a_n)_{n\geq 0}$ is a linear recurrence sequence of order $k \geq 2$ with initial conditions $(a_0, a_1, \ldots, a_{k-1}) = (0, 0, \ldots, 0, 1)$ and recurrence

$$a_{n+k} = c_{k-1}a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n \quad \text{for all} \quad n \geq 0,$$

where $c_0, c_1, \ldots, c_{k-1}$ are integer constants, then

$$a_n = \sum_{i=1}^{k} \frac{\alpha_i^n}{P'(\alpha_i)},$$

where $P(t) = t^k - c_{k-1}t^{k-1} - \cdots - c_1 t - c_0$ is the characteristic polynomial of $(a_n)_{n\geq 0}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of $P(t)$.

If we put $a_n = P_{n-(k-2)}^{(k)}$ for all $n \geq 0$, then we have that $P(t) = \Phi_k(t)$ and

$$h_k'(\gamma_i) = (\gamma_i - 1)\Phi_k'(\gamma_i) \quad \text{for all} \quad 1 \leq i \leq k,$$

where $h_k(x)$ is given by (3.4). So, by using the Kalman's result above we obtain

$$P_n^{(k)} = \sum_{i=1}^{k} \frac{\gamma_i - 1}{(k+1)\gamma_i^2 - 3k\gamma_i + k - 1}\gamma_i^n.$$

This proves the first part Theorem 3.3 (*a*). The above formula is a new way of representing $k-$Pell numbers, but hardly a new result. For instance, Kiliç and Taşci [83] gave another way to do this. But, our formula is perhaps slightly easier to understand, and it also allows us to do some analysis as we will see later.

3.3.2 Properties of the dominant root

First of all, if we consider the function $g_k(x)$ defined in (3.10) as a function of a real variable, then it is not difficult to see that $g_k(x)$ has a vertical asymptote in

$$c(k) := \frac{3k + \sqrt{5k^2+4}}{2(k+1)},$$

and is positive and continuous in $(c(k), +\infty)$. Further,

$$g_k'(x) = -\frac{k\left(x^2-2x+2\right)+(x-1)^2}{\left(k\left(x^2-3x+1\right)+x^2-1\right)^2}$$

is negative in $(c(k), +\infty)$, so $g_k(x)$ is decreasing in $(c(k), +\infty)$. Put

$$a_k = \frac{3k + \sqrt{5k^2+4}}{2(k+1)} + \frac{1}{k} \quad \text{for all} \quad k \geq 2.$$

We next show that the sequence $(a_k)_{k\geq 2}$ is increasing and bounded. To do this, let f be the real function defined by

$$f(x) = \frac{3x + \sqrt{5x^2+4}}{2(x+1)} + \frac{1}{x}.$$

It is then a simple matter to show that

$$f'(x) = \frac{5x-4}{2(x+1)^2\sqrt{5x^2+4}} + \frac{3}{2(x+1)^2} - \frac{1}{x^2} = 0$$

implies that $4(x+1)^2(5x^4 - 10x^3 + 3x^2 - 8x - 4) = 0$. Thus, f has a critical point at $x_0 = 2.14813\ldots$ and is increasing in $[x_0, \infty)$. This, of course, tells us that $(a_k)_{k\geq 2}$ is an increasing sequence. In addition, note that

$$a_k = \frac{3}{2} + \frac{1}{2}\sqrt{5 - \frac{10k+1}{k^2+2k+1}} - \frac{3}{2(k+1)} + \frac{1}{k} \leq \frac{3}{2} + \frac{\sqrt{5}}{2} + \frac{1}{2},$$

which implies that $(a_k)_{k\geq 2}$ is bounded. Additionally,

$$\lim_{k\to\infty} a_k = \varphi^2,$$

where φ denotes the golden section, as usual. Consequently, $a_k \leq \varphi^2$ for all $k \geq 2$ and so $c(k) \leq \varphi^2 - 1/k$ for all $k \geq 2$.

Finally, taking into account that $k < \varphi^{k-2}$ for all $k \geq 6$, it is easy to see that $\varphi^2 - 1/k < \varphi^2(1-\varphi^{-k})$ for all $k \geq 6$.

We summarize what we have proved so far in the following lemma.

Lemma 3.1. *Let $k \geq 2$ be an integer. Then*

(*a*) *The function $g_k(x)$ is positive, decreasing and continuous in the interval $(c(k), \infty)$ and $g_k(x)$ has a vertical asymptote in $c(k)$.*

(*b*) *If $k \geq 2$, then $c(k) \leq \varphi^2 - 1/k$. In addition, if $k \geq 6$, then the inequalities*

$$c(k) \leq \varphi^2 - 1/k < \varphi^2(1-\varphi^{-k})$$

hold.

Recall that each choice of k produces a distinct $k-$generalized Pell sequence which in turn has an associated dominant root $\gamma(k)$. For the convenience of the reader, let us denote by $(\gamma(k))_{k\geq 2}$ the sequence of the dominant roots of the $k-$Pell family of sequences.

Next, we prove that this dominant root is strictly increasing as k increases. We also prove that this dominant root approaches φ^2 as k approaches infinity, and it is larger than $\varphi^2(1-\varphi^{-k})$. The rest of the statements of the following lemma are some technical results which will be used later.

Lemma 3.2. *Let $k, \ell \geq 2$ be integers. Then*

(*a*) *If $k > \ell$, then $\gamma(k) > \gamma(\ell)$.*

(*b*) $\varphi^2(1-\varphi^{-k}) < \gamma(k) < \varphi^2$.

(*c*) *If $k \geq 6$, then*
$$c(k) \leq \varphi^2 - 1/k < \varphi^2(1-\varphi^{-k}) < \gamma(k) < \varphi^2.$$

(*d*) $g_k(\varphi^2) = 1/(\varphi+2)$.

(*e*) $0.276 < g_k(\gamma(k)) < 0.5$ *and* $|g_k(\varphi_i)| < 1$ *for* $2 \leq i \leq k$.

(*f*) $g_k(\gamma)$ *is not an algebraic integer for all* $k \geq 2$.

Before proving this, we note, as an immediate consequence of the preceding lemma, that

$$\lim_{k\to\infty} \gamma(k) = \varphi^2.$$

Proof. To prove (a) we proceed by contradiction by assuming that $\gamma(k) \le \gamma(\ell)$; hence $1/\gamma(\ell)^i \le 1/\gamma(k)^i$ holds for all $i \ge 1$. Taking into account that $\Phi_\ell(\gamma(\ell)) = 0$, one has that

$$(\gamma(\ell))^\ell = 2(\gamma(\ell))^{\ell-1} + (\gamma(\ell))^{\ell-2} + \cdots + \gamma(\ell) + 1,$$

and, of course, the same conclusion remains valid for $\gamma(k)$. From this, we get that

$$\begin{aligned} 1 &= \frac{2}{\gamma(\ell)} + \frac{1}{(\gamma(\ell))^2} + \frac{1}{(\gamma(\ell))^3} + \cdots + \frac{1}{(\gamma(\ell))^\ell} \\ &< \frac{2}{\gamma(k)} + \frac{1}{(\gamma(k))^2} + \frac{1}{(\gamma(k))^3} + \cdots + \frac{1}{(\gamma(k))^k} = 1, \end{aligned}$$

which is a contradiction.

We next prove (b). First, we rewrite the polynomial function (3.4) as

$$h_k(x) = x^{k-1}(x^2 - 3x + 1) + 1. \tag{3.12}$$

Notice that φ^2 is a root of $x^2 - 3x + 1$ because φ is a root of $x^2 - x - 1$. It then follows from (3.12) that $h_k(\varphi^2) = 1$ and therefore $\Phi_k(\varphi^2) = 1/(\varphi^2 - 1) = 1/\varphi > 0$. Since $\Phi_k(2) = 1 - 2^{k-1} < 0$ and recalling that $\Phi_k(x)$ has just one positive real zero we find that $2 < \gamma < \varphi^2$.

On the other hand, by using once more the fact that φ^2 is a root of $x^2 - 3x + 1$ and evaluating the polynomial function (3.12) at γ, we get the relations

$$\varphi^4 - 3\varphi^2 + 1 = 0 \quad \text{and} \quad \gamma^2 - 3\gamma + 1 = -1/\gamma^{k-1}.$$

Subtracting the above equations and rearranging some terms, one obtains

$$(\varphi^2 - \gamma)(\varphi^2 + \gamma - 3) = 1/\gamma^{k-1}.$$

From this and using the facts that $\varphi^2 + \gamma - 3 > 1/\varphi$ and $\varphi < \gamma$, which are easily seen, we get that $\varphi^2 - \gamma < \varphi^2/\varphi^k$ and so $\varphi^2(1 - \varphi^{-k}) < \gamma$. This finishes the proof of (b).

The proof of (c) is a direct consequence of (b) and Lemma 3.1 (b). To prove (c), we observe from (3.10) that

$$g_k\left(\varphi^2\right) = \frac{\varphi^2 - 1}{(k+1)\varphi^4 - 3k(\varphi + 1) + k - 1} = \frac{\varphi}{3\varphi + 1} = \frac{1}{\varphi + 2},$$

where we used the facts that φ and φ^2 are roots of $x^2 - x - 1$ and $x^2 - 3x + 1$, respectively.

We now prove (e). Using (c), (d) and the fact that $g_k(x)$ is decreasing in the interval $(c(k), \infty)$, we have

$$\frac{1}{\varphi+2} = g_k(\varphi^2) < g_k(\gamma(k)) < g_k\left(\varphi^2 - 1/k\right).$$

But

$$\begin{aligned} g_k(\varphi^2 - 1/k) &= \frac{\varphi^2 - \dfrac{1}{k} - 1}{(k+1)\left(\varphi^2 - \frac{1}{k}\right)^2 - 3k\left(\varphi^2 - \frac{1}{k}\right) + k - 1} \\ &= \frac{\varphi - \dfrac{1}{k}}{\varphi - \frac{2\varphi}{k} - \frac{1}{k} + \frac{1}{k^2} + 2} \\ &< \frac{\varphi}{\varphi - \dfrac{2\varphi}{k} - \dfrac{1}{k} + \dfrac{1}{k^2} + 2} \\ &< 0.5, \end{aligned}$$

where the last inequality holds for all $k \geq 11$. Hence, $0.276 < g_k(\gamma(k)) < 0.5$ holds for all $k \geq 11$. Finally, computationally we get that which shows that $0.276 < g_k(\gamma(k)) < 0.5$ also holds for all $k \geq 2$ (see, Table 3.2).

For the second part of (e), we evaluate the polynomial function (3.12) at γ_i and rearranging some terms of the resulting expression, we get the relation

$$\gamma_i^2 - 3\gamma_i + 1 = -1/\gamma_i^{k-1},$$

and so

$$k(\gamma_i^2 - 3\gamma_i + 1) + \gamma_i^2 - 1 = \gamma_i^2 - 1 - \frac{k}{\gamma_i^{k-1}}.$$

Hence,

$$|k(\gamma_i^2 - 3\gamma_i + 1) + \gamma_i^2 - 1| = \left|\frac{k}{\gamma_i^{k-1}} - (\gamma_i^2 - 1)\right| \geq \frac{k}{|\gamma_i|^{k-1}} - |\gamma_i^2 - 1| > k - 2,$$

where we used the fact that $|\gamma_i| < 1$ because $2 \leq i \leq k$. Consequently,

$$|g_k(\gamma_i)| = \frac{|\gamma_i - 1|}{|k(\gamma_i^2 - 3\gamma_i + 1) + \gamma_i^2 - 1|} < \frac{2}{k-2} \leq 1 \quad \text{for all} \quad k \geq 4.$$

The cases $k = 2$ and 3 can be checked computationally.

For the proof of (f), assume that $g_k(\gamma)$ is an algebraic integer. Then its norm (from $\mathbb{Q}(\alpha)$ to $\mathbb{Q}$) is an integer. Applying the norm and taking absolute values, we obtain

$$1 \leq \left|N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(g_k(\gamma))\right| = g_k(\gamma)\prod_{i=2}^{k}|g_k(\gamma_i)|.$$

However, $g_k(\gamma) < 0.5$ and $|g_k(\gamma_i)| < 2/(k-2) \leq 1$ for $i = 2,\ldots,k$ and all $k \geq 4$, contradicting the above inequality. The case $k = 2,3$ are clear. This finishes the proof of the lemma. $\square$

Table 3.2: First values of $g_k(\gamma(k))$

k	2	3	4	5	6	7	8	9	10
$g_k(\gamma(k))$	0.35...	0.30...	0.29...	0.28...	0.27...	0.27...	0.27...	0.27...	0.27...

3.3.3 Sequence of errors

For an fixed integer $k \geq 2$ and $n \geq 2-k$, define $E_n^{(k)}$ to be the error of the approximation of the nth $k-$Pell number with the dominant term of the Binet–style formula of $P^{(k)}$ given in Theorem 3.3 (a), i.e.,

$$E_n^{(k)} = P_n^{(k)} - g_k(\gamma)\gamma^n, \tag{3.13}$$

for γ the dominant root of $\Phi(x)$ and $g_k(x)$ defined as in (3.10).

Given a polynomial f, it is well known that the set of all possible linear recurrence sequences of real numbers having the characteristic equation $f(x) = 0$ is a vector space over the real numbers. Since $P^{(k)}$ and $(\gamma^n)_{n\geq 2}$ satisfy the characteristic equation $\Phi_k(x) = 0$, it follows from (3.13) that the sequence $(E_n^{(k)})_{n\geq 2-k}$ satisfies the same recurrence relation as the $k-$Pell sequence. We record this as follows.

Lemma 3.3. *Let $k \geq 2$ be an integer. Then*

$$E_n^{(k)} = 2E_{n-1}^{(k)} + E_{n-2}^{(k)} + \cdots + E_{n-k}^{(k)} \quad \textit{for all} \quad n \geq 2.$$

Furthermore, if $n \geq 3$, then

$$E_n^{(k)} = 3E_{n-1}^{(k)} - E_{n-2}^{(k)} - E_{n-k-1}^{(k)}.$$

The following and last result of this section shows that the error of the approximation defined in (3.13) is eventually zero, which yields that $g_k(\gamma)\gamma^n$ provides a good approximation of $P_n^{(k)}$ for every sufficiently large n.

Lemma 3.4. *For an fixed integer $k \geq 2$ we have*

$$\lim_{n\to\infty} E_n^{(k)} = 0.$$

Proof. Using the fact that $\lim_{n\to\infty} |\gamma_j|^n = 0$ for all $2 \leq j \leq k$ and taking into account that

$$|E_n^{(k)}| \leq \sum_{j=2}^{k} |g_k(\gamma_j)||\gamma_j|^n,$$

we deduce that

$$\lim_{n\to\infty} |E_n^{(k)}| = 0.$$

This proves the lemma. □

To conclude this subsection, we prove the second part of Theorem 3.3 (a). Indeed, with the notation above, we have to prove that

$$|E_n^{(k)}| < 1/2 \quad \text{for all} \quad k \geq 2 \quad \text{and} \quad n \geq 2-k.$$

In order to do this, we proof three claims.

Claim 1: $|E_n^{(k)}| < 1/2 \quad \text{for all} \quad 2-k \leq n \leq 0.$

Proof. Because the initial conditions of $P^{(k)}$, we have that $P_n^{(k)} = 0$ for all $2-k \leq n \leq 0$, so $E_n^{(k)} = -g_k(\gamma)\gamma^n$ for all $2-k \leq n \leq 0$. For the case $n = 0$, we have, by Lemma 3.2 (e), that $|E_0^{(k)}| = g_k(\gamma) < 0.5$. If $2-k \leq n \leq -1$, then $\gamma^n < \gamma^{-1} < 1$ and therefore $g_k(\gamma)\gamma^n < g_k(\gamma) < 1/2$ for all $k \geq 2$. □

Claim 2: $|E_1^{(k)}| < 1/2.$

Proof. First, note that $E_1^{(k)} = P_1^{(k)} - g_k(\gamma)\gamma = 1 - g_k(\gamma)\gamma$. By Lemma 3.2 (e) we have that $0.276\gamma < g_k(\gamma)\gamma < 0.5\gamma$. However, $0.66 < 0.276\gamma(2) \leq 0.276\gamma$ and $0.5\gamma < 0.5\varphi^2 < 1.31$, and so $0.66 < g_k(\gamma)\gamma < 1.31$. Thus, $-0.31 < 1 - g_k(\gamma)\gamma < 0.34$ which implies that $|E_1^{(k)}| = |1 - g_k(\gamma)| < 1/2$ for all $k \geq 2$. □

Claim 3: $|E_n^{(k)}| < 1/2 \quad \text{for all} \quad n \geq 2.$

Proof. Suppose for the sake of contradiction that $|E_n^{(k)}| \geq 1/2$ for some integer $n \geq 2$, and let n_0 be the smallest positive integer such that $|E_{n_0}^{(k)}| \geq 1/2$. Since $|E_{n_0-1}^{(k)}| < 1/2$ and $|E_{n_0-k}^{(k)}| < 1/2$ we get $|E_{n_0-1}^{(k)} + E_{n_0-k}^{(k)}| < 1$. According to Lemma 3.3

$$E_{n_0+1}^{(k)} = 3E_{n_0}^{(k)} - (E_{n_0-1}^{(k)} + E_{n_0-k}^{(k)}),$$

and so

$$|E_{n_0+1}^{(k)}| \geq 3|E_{n_0}^{(k)}| - |E_{n_0-1}^{(k)} + E_{n_0}^{(k)}|.$$

Hence

$$|E_{n_0+1}^{(k)}| - |E_{n_0}^{(k)}| \geq 2|E_{n_0}^{(k)}| - |E_{n_0-1}^{(k)} + E_{n_0-k}^{(k)}| > 0$$

giving

$$|E_{n_0+1}^{(k)}| > |E_{n_0}^{(k)}|.$$

Since $n_0 - k + 1 < n_0$ we infer that $|E_{n_0-k+1}^{(k)}| < 1/2 \leq |E_{n_0}^{(k)}| < |E_{n_0+1}^{(k)}|$ and therefore $|E_{n_0}^{(k)} + E_{n_0-k+1}^{(k)}| < 2|E_{n_0+1}^{(k)}|$. By using this and Lemma 3.3, we obtain

$$|E_{n_0+2}^{(k)}| \geq 3|E_{n_0+1}^{(k)}| - |E_{n_0}^{(k)} + E_{n_0-k+1}^{(k)}| > 3|E_{n_0+1}^{(k)}| - 2|E_{n_0+1}^{(k)}|.$$

Hence, $|E_{n_0+2}^{(k)}| > |E_{n_0+1}^{(k)}|$.

Now suppose that $|E_{n_0}^{(k)}| < |E_{n_0+1}^{(k)}| < \cdots < |E_{n_0+i-1}^{(k)}|$ for some integer $i \geq 4$. We distinguish two cases according to whether $n_0 + i - k - 1 < n_0$ or $n_0 \leq n_0 + i - k - 1$.

First, if $n_0 + i - k - 1 < n_0$, then we get

$$|E_{n_0+i-k-1}^{(k)}| < 1/2 \leq |E_{n_0}^{(k)}| < |E_{n_0+1}^{(k)}| < \cdots < |E_{n_0+i-1}^{(k)}|.$$

If $n_0 \leq n_0 + i - k - 1 < n_0 + i - 1$, then we obtain that $|E_{n_0+i-k-1}^{(k)}| < |E_{n_0+i-1}^{(k)}|$.

In any case, we have that the inequality

$$|E_{n_0+i-k-1}^{(k)}| < |E_{n_0+i-1}^{(k)}|$$

always holds. For this reason

$$|E_{n_0+i-2}^{(k)} + E_{n_0+i-k-1}^{(k)}| < 2|E_{n_0+i-1}^{(k)}|.$$

From Lemma 3.3 once more, we have that $E^{(k)}_{n_0+i} = 3E^{(k)}_{n_0+i-1} - (E^{(k)}_{n_0+i-2} + E^{(k)}_{n_0+i-k-1})$ and so

$$\begin{aligned} |E^{(k)}_{n_0+i}| &\geq 3|E^{(k)}_{n_0+i-1}| - |E^{(k)}_{n_0+i-2} + E^{(k)}_{n_0+i-k-1}| \\ &> 3|E^{(k)}_{n_0+i-1}| - 2|E^{(k)}_{n_0+i-1}| \\ &= |E^{(k)}_{n_0+i-1}|. \end{aligned}$$

Consequently,

$$|E^{(k)}_{n_0+i}| > |E^{(k)}_{n_0+i-1}| > \cdots > |E^{(k)}_{n_0+1}| > |E^{(k)}_{n_0}|,$$

contradicting Lemma 3.4 which says that the error must eventually go to 0. □

The proof of the second part of Theorem 3.3 (a) is a direct consequence of the above three claims.

3.3.4 Exponential growth

We begin by mentioning that for the Fibonacci sequence and the Pell sequence, it is well–known that

$$\varphi^{n-2} \leq F_n \leq \varphi^{n-1} \quad \text{holds for all} \quad n \geq 1, \tag{3.14}$$

and

$$\gamma^{n-2} \leq P_n \leq \gamma^{n-1} \quad \text{holds for all} \quad n \geq 1. \tag{3.15}$$

This exhibits an exponential growth of the Fibonacci and Pell numbers. In the above, the value of γ is $\gamma(2) = 1 + \sqrt{2}$. We finally prove Theorem 3.3 (b) which shows that the above inequality (3.15) holds for the $k-$Pell sequence as well. This will be done by using mathematical induction on n.

To begin with, we show that inequality (3.11) holds for $n = 1, 2, \ldots, k$. It is clear that the result is true for $n = 1$ because $\gamma > 1$. For $n = 2, \ldots, k$ we know, by (3.3), that $P^{(k)}_n = F_{2n-1}$, so we need to show that

$$\gamma^{n-2} \leq F_{2n-1} \leq \gamma^{n-1} \quad \text{for} \quad 2 \leq n \leq k. \tag{3.16}$$

By Lemma 3.2 (b) and (3.14), we get

$$\gamma^{n-2} < \varphi^{2(n-2)} < \varphi^{2n-3} \leq F_{2n-1}$$

and therefore the left–hand side of the above inequality (3.16) holds. Then, it remains to prove that

$$F_{2n-1} \leq \gamma^{n-1} \quad \text{holds for} \quad 2 \leq n \leq k. \tag{3.17}$$

Then, by direct inspection one checks that the inequality (3.17) holds true for $2 \le k \le 6$, so we may assume that $k \ge 7$. Now, by making use of (2.6), we get

$$F_{2n-1} = \frac{\varphi^{2n-1} + \varphi^{-(2n-1)}}{\sqrt{5}} = \frac{\varphi^{2n-1}}{\sqrt{5}}\left(1 + \frac{1}{\varphi^{4n-2}}\right).$$

Since $\varphi^{2(n-1)}(1-\varphi^{-k})^{n-1} < \gamma^{n-1}$ because $\varphi^2(1-\varphi^{-k}) < \gamma$ by Lemma 3.2 (b), it suffices to prove that

$$\frac{\varphi^{2n-1}}{\sqrt{5}}\left(1 + \frac{1}{\varphi^{4n-2}}\right) \le \varphi^{2(n-1)}(1-\varphi^{-k})^{n-1},$$

which is equivalent to

$$1 + \frac{1}{\varphi^{4n-2}} \le \frac{\sqrt{5}}{\varphi}(1-\varphi^{-k})^{n-1}. \tag{3.18}$$

Using the fact that the function $x \to (1-\varphi^{-x})^{x-1}$ is increasing for $x \ge 7$ and taking into account that $2 \le n \le k$ and $k \ge 7$, we deduce that

$$1 + \frac{1}{\varphi^{4n-2}} \le 1 + \frac{1}{\varphi^6} = 1.05572\ldots,$$

whereas

$$\frac{\sqrt{5}}{\varphi}(1-\varphi^{-k})^{n-1} \ge \frac{\sqrt{5}}{\varphi}(1-\varphi^{-k})^{k-1} \ge \frac{\sqrt{5}}{\varphi}(1-\varphi^{-7})^{6} = 1.11987\ldots.$$

This proves inequality (3.18). Thus, we have proved that inequality (3.11) holds for the first k nonzero terms of $P^{(k)}$.

Finally, suppose that (3.11) holds for all terms $P_m^{(k)}$ with $m \le n-1$ for some $n > k$. It then follows from the recurrence relation of $P^{(k)}$ that

$$2\gamma^{n-3} + \gamma^{n-4} + \cdots + \gamma^{n-k-2} \le P_n^{(k)} \le 2\gamma^{n-2} + \gamma^{n-3} + \cdots + \gamma^{n-k-1}.$$

So

$$\gamma^{n-k-2}(2\gamma^{k-1} + \gamma^{k-2} + \cdots + 1) \le P_n^{(k)} \le \gamma^{n-k-1}(2\gamma^{k-1} + \gamma^{k-2} + \cdots + 1),$$

which combined with the fact that $\gamma^k = 2\gamma^{k-1} + \gamma^{k-2} + \cdots + 1$ gives the desired result. Thus, inequality (3.11) holds for all positive integers n. So, the proof of Theorem 3.3 is now complete.

Chapter 4

Combinatorial Interpretation of generalized Pell numbers

In this chapter, we give combinatorial interpretations for the $k-$generalized Pell sequence by means of lattice paths and generalized bi–colored compositions. We also derive some basic relations and identities by using Riordan arrays.

4.1 Introduction

There are a lot of integer sequences which are used in almost every field of modern sciences. For instance, the Fibonacci sequence $F := (F_n)_{n\geq 0}$ is one of the most famous and curious numerical sequences in mathematics and has been widely studied in the literature. The Fibonacci numbers can be interpreted combinatorially as the number of ways to tile a board of length n and height 1 using only squares (length 1, height 1) and dominoes (length 2, height 1). They also count the number of binary sequences with no consecutive zeros, the number of sequences of 1's and 2's which sum to a given number, the number of independent sets of a path graph, among others.

Like the Fibonacci numbers, the $k-$Fibonacci numbers can also be interpreted combinatorially as the number of ways to tile a board of length n and height 1 using now tiles of length at most k. This combinatorial interpretation has been used to provide simple and intuitive proofs of several identities involving $k-$generalized Fibonacci numbers (see

[69]). On the other hand, Bernini [14] provides, via a simple bijection, some interesting relations involving $k-$Fibonacci numbers with the set of length n binary strings avoiding k of consecutive 0's, the set of length n strings avoiding $k+1$ consecutive 0's, and 1's with some more restriction on the first and last letter. In 2005, Egge and Mansour [53] extended a work of Simion and Schmidt [122] by showing that the set of permutations avoiding the patterns `12...k`, `132` and `213` is counted by the $(k-1)-$Fibonacci numbers. In later years, Juarna and Vajnovszki [77] generalized Egge and Mansour's work.

The Pell sequence has also many interesting combinatorial properties similar to those known for the Fibonacci sequence (see Koshy's books [87, 88]). For instance, it is possible to prove that P_{n+1} counts the number of bi–colored compositions of a positive integer n. By a *bi–colored composition* of a positive integer n we mean a sequence of positive integers $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_\ell)$ such that $\sigma_1 + \sigma_2 + \cdots + \sigma_\ell = n$, $\sigma_i \in \{1, 2\}$, and the summand 1 can come in one of two different colors. The colors of the summand 1 are denoted by subscripts 1_1 and 1_2. For example, the bi–colored compositions of 3 are

$$2+1_1,\quad 2+1_2,\quad 1_1+2,\quad 1_2+2,\quad 1_1+1_1+1_1,\quad 1_2+1_1+1_1,\quad 1_1+1_2+1_1,$$
$$1_1+1_1+1_2,\quad 1_1+1_2+1_2,\quad 1_2+1_1+1_2,\quad 1_2+1_2+1_1,\quad 1_2+1_2+1_2.$$

This combinatorial interpretation can be translated into the language of tilings. As mentioned before, it is well–known that the Fibonacci number F_{n+1} can be interpreted as the number of tilings of a board of length n with cells labeled 1 to n from left to right with only squares and dominoes [13]. If we use white and black squares and non–colored dominoes we obtain a different combinatorial interpretation for the Pell numbers. For example, Figure 4.1 shows the different ways to tiling a $3-$board.

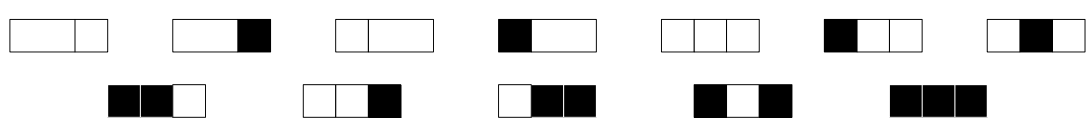


Figure 4.1: Different ways to tile $3-$boards.

In this chapter, we introduce new combinatorial interpretations for the $k-$Pell sequence by means of lattice paths and generalized bi–colored compositions. We also use Riordan arrays to derive possibly new combinatorial identities and relations for the $k-$Pell numbers.

4.2 A combinatorial interpretation: lattice paths

Let S be a fixed subset of $\mathbb{Z}\times\mathbb{Z}$. A *lattice path* Γ of length ℓ with steps in S is a $\ell-$tuple of directed steps of S. That is $\Gamma = (s_1, \dots, s_\ell)$ where $s_i \in S$ for $1 \le i \le \ell$. Let $a(n,m)$ be the number of lattice paths from the point $(0,0)$ to the point (n,m) with step set $S = \{H = (1,0), V = (0,1)\}$. It is clear that

$$a(n,m) = \binom{n+m}{n}.$$

Let $\mathcal{A}$ be the infinite lower triangular matrix defined by

$$\mathcal{A} \coloneqq [a(n-m,m)]_{n,m\ge 0} = \left[\binom{n}{m}\right]_{n,m\ge 0}.$$

The matrix $\mathcal{A}$ coincides with the Pascal matrix. Among the many properties of the Pascal matrix, it is known that the sum of the elements on the rising diagonal is the Fibonacci sequence, i.e., for $n \ge 1$

$$F_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i}.$$

From this combinatorial interpretation, we conclude that F_n counts the number of lattice paths from $(0,0)$ to $(n-2i-1, i)$ for $i = 0, 1, \dots, \lfloor (n-1)/2 \rfloor$. For example, Figure 4.2 shows the paths for $n = 5$, i.e., the paths counted by the Fibonacci number $F_5 = 5$.

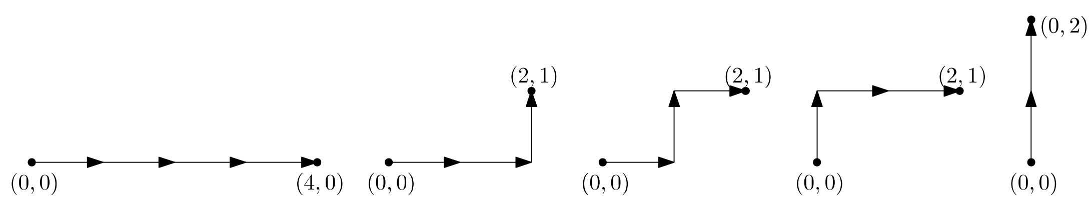


Figure 4.2: Lattices paths counted by the Fibonacci number F_5.

The goal of this section is to generalize the above results for the $k-$Pell numbers. In particular, we introduce a family of matrices $\mathcal{P}_k$ from a family of generalized paths. These matrices satisfy that the row sum coincides with the $k-$Pell numbers; see Corollary 4.2.

Let $\mathbb{P}_k(n,m)$ denote the set of lattice paths from the point $(0,0)$ to the point (n,m) with step set

$$S_k \coloneqq \{H = (1,0),\ V = (0,1),\ D_1 = (1,1),\ D_2 = (1,2),\ \dots,\ D_k = (1,k)\}.$$

In Figure 4.3, we show all lattice paths of the set $\mathbb{P}_2(1,3)$.

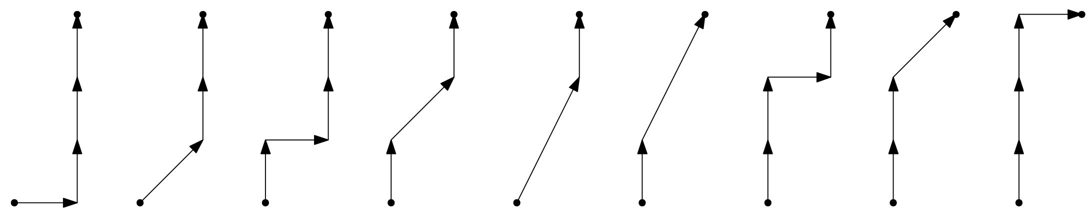
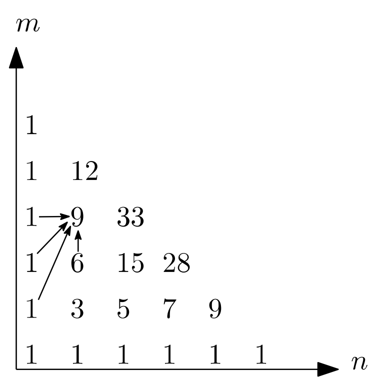


Figure 4.3: Lattices paths in $\mathbb{P}_2(1,3)$.

Let $p_k(n,m)$ be the number of lattice paths of $\mathbb{P}_k(n,m)$, i.e., $p_k(n,m) := |\mathbb{P}_k(n,m)|$. Since the last step on any path from $\mathbb{P}_k(n,m)$ is one of S_k, we obtain the recurrence relation:

$$\begin{aligned} p_k(n,m) = p_k(n-1,m) + p_k(n,m-1) + p_k(n-1,m-1) \\ + p_k(n-1,m-2) + \cdots + p_k(n-1,m-k), \end{aligned} \quad (4.1)$$

with $n \geq 1, m \geq k$, and the initial conditions $p_k(0,m) = 1 = p_k(n,0)$. For example, for $k = 2$ the first few values of the sequence $p_2(n,m)$ are



Let $P_n^{(k)}(x)$ be the ordinary generating function of the sequence $(p_k(n,m))_{m\geq 0}$. That is,

$$P_n^{(k)}(x) = \sum_{i\geq 0} p_k(n,i)x^i.$$

In Theorem 4.1 we find an expression for the generating function $P_n^{(k)}(x)$.

Theorem 4.1. *We have*

$$P_n^{(k)}(x) = \frac{(1+x+x^2+\cdots+x^k)^n}{(1-x)^{n+1}}.$$

Proof. From equation (4.1), we obtain the relation

$$P_n^{(k)}(x) = P_{n-1}^{(k)}(x) + xP_n^{(k)}(x) + xP_{n-1}^{(k)}(x) + x^2P_{n-1}^{(k)}(x) + \cdots + x^kP_{n-1}^{(k)}(x).$$

Thus

$$P_n^{(k)}(x) = \frac{1 + x + x^2 + \cdots + x^k}{1-x}P_{n-1}^{(k)}(x).$$

Since $P_0 = 1/(1-x)$, we obtain the desired result. □

Corollary 4.1. *The number of lattice paths $p_k(n,m)$ is given by*

$$p_k(n,m) = \sum_{\ell_0+\ell_1+\cdots+\ell_k=n} \binom{n}{\ell_0, \ell_1, \ldots, \ell_k}\binom{n+m-t}{n},$$

where $t = \sum_{s=0}^k s\ell_s$ and

$$\binom{n}{n_1, \ldots, n_m} = \frac{n!}{n_1! \ \cdots \ n_m!}$$

is the multinomial coefficient.

Proof. From the multinomial theorem, the generating function

$$\frac{1}{(1-x)^{n+1}} = \sum_{i\geq 0}\binom{n+i}{i}x^i,$$

and Theorem 4.1, we have that

$$\begin{aligned} p_k(n,m) &= [x^m]P_n^{(k)}(x) = [x^m]\frac{(1+x+x^2+\cdots+x^k)^n}{(1-x)^{n+1}} \\ &= [x^m]\sum_{\ell_0+\ell_1+\cdots+\ell_k=n}\binom{n}{\ell_0, \ell_1, \ldots, \ell_k}\prod_{s=0}^k x^{s\ell_s}\sum_{i\geq 0}\binom{n+i}{i}x^i \\ &= [x^m]\sum_{\ell_0+\ell_1+\cdots+\ell_k=n}\sum_{i\geq 0}\binom{n}{\ell_0, \ell_1, \ldots, \ell_k}\binom{n+i}{i}x^{t+i}, \end{aligned}$$

where $t = \sum_{s=0}^k s\ell_s$. By comparing the $m-$th coefficient we obtain the desired result. □

For example,

$$\begin{aligned} p_2(1,3) &= \sum_{\ell_0+\ell_1+\ell_2=1}\binom{1}{\ell_0, \ell_1, \ell_2}\binom{1+3-(\ell_1+2\ell_2)}{1} \\ &= \binom{1}{1,0,0}\binom{4}{1} + \binom{1}{0,1,0}\binom{3}{1} + \binom{1}{0,0,1}\binom{2}{1} = 4+3+2 = 9. \end{aligned}$$

In Figure 4.3, we show the corresponding lattice paths.

Let $\mathcal{P}_k := [q_k(n,m)]_{n,m\geq 0}$ be the array defined by

$$q_k(n,m) = \begin{cases} p_k(m, n-m), & \text{if } n \geq m; \\ 0, & \text{if } n < m. \end{cases}$$

For example, the first few rows of the array $\mathcal{P}_2$ are as follows (see sequence A102036 in [123]).

$$\mathcal{P}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 9 & 15 & 7 & 1 & 0 & 0 & 0 \\ 1 & 12 & 33 & 28 & 9 & 1 & 0 & 0 \\ 1 & 15 & 60 & 81 & 45 & 11 & 1 & 0 \\ 1 & 18 & 96 & 189 & 161 & 66 & 13 & 1 \\ \vdots & & & \vdots & & & & \vdots \end{pmatrix}$$

This new family of matrices $\mathcal{P}_k$ are an example of a Riordan array. Remember that an infinite lower triangular matrix is called a *Riordan array* [120] if its kth column satisfies the generating function $g(x)\left(f(x)\right)^k$ for $k \geq 0$, where $g(x)$ and $f(x)$ are formal power series with $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$. The matrix corresponding to the pair $f(x), g(x)$ is denoted by $(g(x), f(x))$. If we multiply (g, f) by a column vector $(c_0, c_1, \dots)^T$ with the generating function $h(x)$, then the resulting column vector has generating function $g(x)h(f(x))$. This property is known as the fundamental theorem of Riordan arrays or summation property.

The product of two Riordan arrays $(g(x), f(x))$ and $(h(x), l(x))$ is defined by

$$(g(x), f(x)) * (h(x), l(x)) = \left(g(x)h\left(f(x)\right), l\left(f(x)\right)\right).$$

We recall that the set of all Riordan matrices is a group under the operator " $*$ " [120]. The identity element is $I = (1, x)$, and the inverse of $(g(x), f(x))$ is

$$(g(x), f(x))^{-1} = \left(1/\left(g \circ \overline{f}\right)(x), \overline{f}(x)\right),$$

where $\overline{f}(x)$ is the compositional inverse of $f(x)$. For example, the Pascal matrix is given by the Riordan array

$$\left(\frac{1}{1-x}, \frac{x}{1-x}\right).$$

Several authors have used Riordan arrays to study lattice paths; see for example [49, 59, 98, 111, 112, 124, 137, 138, 139, 140].

From the definition of Riordan array and Theorem 4.1 we obtain the following theorem.

Theorem 4.2. *The matrix $\mathcal{P}_k$ is a Riordan array given by*

$$\mathcal{P}_k = \left(\frac{1}{1-x}, x\frac{1+x+x^2+\cdots+x^k}{1-x}\right).$$

Proof. The $(n,m)-$th entry of the Riordan array is given by

$$\begin{aligned}[x^n]\frac{1}{1-x}\left(x\frac{1+x+x^2+\cdots+x^k}{1-x}\right)^m &= [x^{n-m}]\frac{(1+x+x^2+\cdots+x^k)^m}{(1-x)^{m+1}}\\ &= [x^{n-m}]P_m^{(k)}(x)\\ &= p_k(m,n-m) = q_k(n,m).\end{aligned}$$

Hence the matrices are the same. □

Let $R_k(x)$ be the generating function for the rows sums of the matrix $\mathcal{P}_k$. In Theorem 4.3 we give a generating function for $R_k(x)$.

Theorem 4.3. *The generating function $R_k(x)$ is given by*

$$R_k(x) = \frac{1}{1-2x-x^2-\cdots-x^{k+1}}.$$

Proof. From the summation property for the Riordan arrays we have

$$R_k(x) = \mathcal{P}_k\left(\frac{1}{1-x}\right) = \frac{1}{1-x}\left(\frac{1}{1-x\frac{1+x+x^2+\cdots+x^k}{1-x}}\right) = \frac{1}{1-2x-x^2-\cdots-x^{k+1}}.$$ □

By using standard methods, it is possible to prove that the ordinary generating function of the $k-$Pell sequence is

$$\sum_{n\geq 0} P_n^{(k)}x^n = \frac{1}{1-2x-x^2-\cdots-x^k}.$$

Thus we have the following corollary.

Corollary 4.2. *The $k-$Pell numbers $P_n^{(k)}$ coincide with the row sum of the matrix $\mathcal{P}_{k-1}$.*

For example, the row sum of the matrix $\mathcal{P}_2$ coincides with the 3−Pell numbers (see sequence A077939 in [123]):

$$1,\ 2,\ 5,\ 13,\ 33,\ 84,\ 214,\ 545,\ 1388,\ 3535,\ 9003, \ldots$$

In Corollary 4.3 we deduce a new combinatorial identity for the $k-$Pell numbers.

Corollary 4.3. *The $k-$Pell numbers $P_n^{(k)}$ are given by the combinatorial identity*

$$P_n^{(k)} = \sum_{i=0}^{n} \sum_{\ell_0+\ell_1+\cdots+\ell_{k-1}=i} \binom{i}{\ell_0, \ell_1, \ldots, \ell_{k-1}} \binom{n-t}{i},$$

where $t = \sum_{j=0}^{k-1} j\ell_j$.

Proof. From Corollaries 4.1 and 4.2 we have

$$P_n^{(k)} = \sum_{i=0}^{n} q_{k-1}(n,i) = \sum_{i=0}^{n} p_{k-1}(i, n-i) = \sum_{i=0}^{n} \sum_{\ell_0+\ell_1+\cdots+\ell_{k-1}=i} \binom{i}{\ell_0, \ell_1, \ldots, \ell_{k-1}} \binom{n-t}{i}.$$

□

Finally, from the relation $P_n^{(k)} = \sum_{i=0}^{n} p_{k-1}(i, n-i)$ we deduce the following combinatorial interpretation.

Theorem 4.4. *The $k-$Pell number $P_{n+1}^{(k)}$ counts the number of lattice paths from the point $(0,0)$ to $(n-i, i)$ for $i = 0, 1, \ldots, n$, with step set*

$$S_k = \{H = (1,0),\ V = (0,1),\ D_1 = (1,1),\ D_2 = (1,2),\ \ldots,\ D_k = (1,k)\}.$$

For example, the 3−Pell number $P_4^{(3)} = 13$ counts the paths of Figure 4.4.

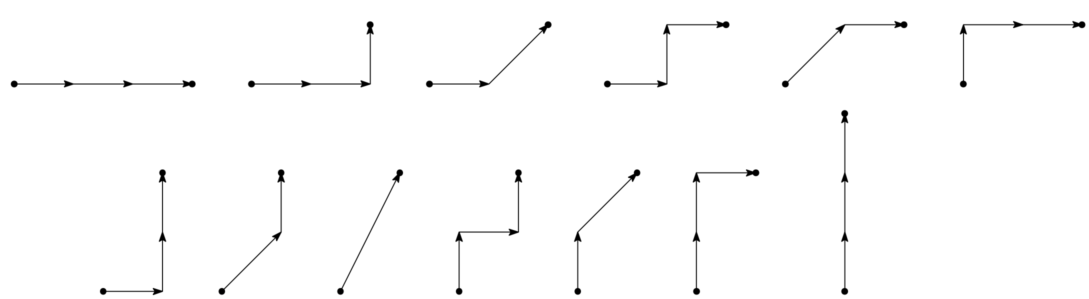


Figure 4.4: Lattices paths counted by $P_4^{(3)}$.

We recall that the Fibonacci numbers are equal to the sum on the rising diagonal in the Pascal matrix. In Theorem 4.5 we give an analogue of this result for the $k-$Pell sequence.

Theorem 4.5. *The $k-$Pell numbers $P_n^{(k)}$ coincide with the sum of the elements on rising diagonal lines in the Riordan array*

$$Q_k := \left(\frac{1}{1-2x}, x\frac{1+x+x^2+\cdots+x^{k-2}}{1-2x}\right).$$

Proof. The generating function of the sum of the elements on rising diagonal lines in the above Riordan array is

$$\frac{1}{1-2x}\left(\frac{1}{1-x^2\left(\frac{1+x+x^2+\cdots+x^{k-2}}{1-2x}\right)}\right) = \frac{1}{1-2x-x^2-\cdots-x^k} = \sum_{n\geq 0} P_n^{(k)}x^n. \qquad \square$$

For example, the diagonal sum of the Riordan array Q_2 (see sequence A038207 in [123]) coincides with the classical Pell numbers

$$Q_2 = \left(\frac{1}{1-2x}, \frac{x}{1-2x}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 8 & 12 & 6 & 1 & 0 & 0 & 0 & 0 \\ 16 & 32 & 24 & 8 & 1 & 0 & 0 & 0 \\ 32 & 80 & 80 & 40 & 10 & 1 & 0 & 0 \\ 64 & 192 & 240 & 160 & 60 & 12 & 1 & 0 \\ 128 & 448 & 672 & 560 & 280 & 84 & 14 & 1 \\ \vdots & & & & \vdots & & & \vdots \end{pmatrix}.$$

The diagonal sum of the Riordan array Q_3 coincides with the $3-$Pell numbers

$$Q_3 = \left(\frac{1}{1-2x}, x\frac{1+x}{1-2x}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 8 & 16 & 8 & 1 & 0 & 0 & 0 & 0 \\ 16 & 44 & 37 & 11 & 1 & 0 & 0 & 0 \\ 32 & 112 & 134 & 67 & 14 & 1 & 0 & 0 \\ 64 & 272 & 424 & 305 & 106 & 17 & 1 & 0 \\ 128 & 640 & 1232 & 1168 & 584 & 154 & 20 & 1 \\ \vdots & & & \vdots & & & & \vdots \end{pmatrix}.$$

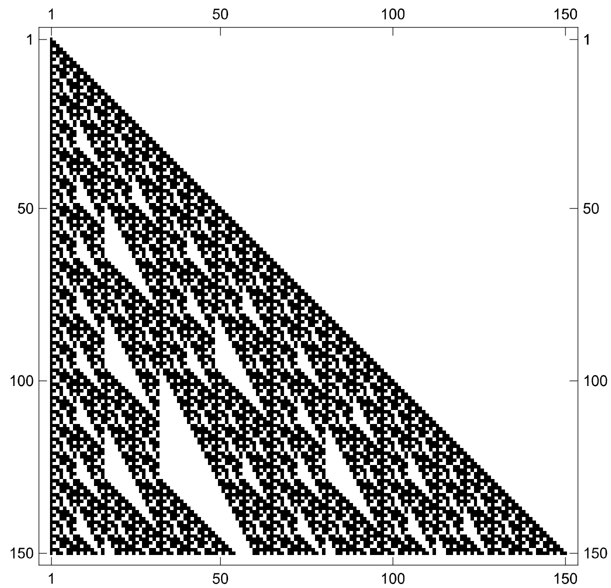


Figure 4.5: Matrix $\mathcal{P}_2$ (mod 2).

The Riordan arrays obtained in this section show interesting patterns if you evaluated their entries mod 2. In Figure 4.5 we show the *fractal* structure of the matrix $\mathcal{P}_2$. Notice that Merlini and Nocentini [105] have studied some relations between Riordan arrays and fractal patterns. In a forthcoming paper we will study the $p-$adic valuation for the $k-$Pell sequence.

4.3 The generalized bi–colored compositions

The goal of this section is to consider a generalization of the concept of a bi-colored composition in order to give another combinatorial interpretation of the $k-$Pell numbers. Here and below, n denotes a positive integer. In fact, we defined a *generalized bi–colored composition* of n as a sequence of positive integers $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_\ell)$ such that $\sigma_1 + \sigma_2 + \cdots + \sigma_\ell = n$, and the summand $\mathtt{1}$ can take two colors. The colors of the summand $\mathtt{1}$ are denoted by subscripts $\mathtt{1}_1$ and $\mathtt{1}_2$. Further, the positive integers σ_i are called *parts* of the composition. We let $\mathbb{A}_n$ denote the set of all generalized bi–colored compositions of n and let $C(n)$ denote the number of elements in $\mathbb{A}_n$, i.e., $C(n) \coloneqq |\mathbb{A}_n|$. We also use $C_k(n)$ to denote the number of generalized bi–colored compositions of n with parts in the set $\{\mathtt{1}, \mathtt{2}, \ldots, \mathtt{k}\}$.

For example,

$$\mathbb{A}_3 = \{3, 2+1_1, 2+1_2, 1_1+2, 1_2+2, 1_1+1_1+1_1, 1_1+1_1+1_2, 1_1+1_2+1_1, \\ 1_1+1_2+1_2, 1_2+1_1+1_1, 1_2+1_1+1_2, 1_2+1_2+1_1, 1_2+1_2+1_2\}.$$

Therefore, $C(3) = 13$. Finally, let $\mathbb{F}_n$ denote the set of classical compositions of n with parts in $\{1, 2\}$. It is well-know that

$$|\mathbb{F}_n| = F_{n+1} \quad \text{for all} \quad n \geq 1.$$

With the above notation, we have the following theorem.

Theorem 4.6. *There is a bijection from* $\mathbb{A}_n$ *to* $\mathbb{F}_{2n}$*. So*

$$|\mathbb{A}_n| = |\mathbb{F}_{2n}| = F_{2n+1} \quad \textit{for all} \quad n \geq 1.$$

Proof. The result clearly holds for $n = 1$, so we assume that $n \geq 2$. We shall define the map φ from $\mathbb{A}_n$ to $\mathbb{F}_{2n}$ as follows:

$$(1_1) \longmapsto (1, 1), \qquad (1_2) \longmapsto (2),$$

$$(2) \longmapsto (1, 2, 1), \qquad (3) \longmapsto (1, 2, 2, 1), \qquad \ldots \qquad , (n) \longmapsto (1, \underbrace{2, \ldots, 2}_{(n-1)\text{-times}}, 1)$$

For every composition $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_\ell)$ in $\mathbb{A}_n$, we define

$$\varphi(\sigma) = (\varphi(\sigma_1), \varphi(\sigma_2), \ldots, \varphi(\sigma_\ell)).$$

For example,

$$\varphi(3, 1_2, 2, 2, 1_1, 4) = (1, 2, 2, 1, 2, 1, 2, 1, 1, 2, 1, 1, 1, 1, 2, 2, 2, 1).$$

Note that if $\sigma \in \mathbb{A}_n$, then $\varphi(\sigma)$ is a composition of $2n$ with parts in $\{1, 2\}$, i.e., $\varphi(\sigma) \in \mathbb{F}_{2n}$ for all $\sigma \in \mathbb{A}_n$. Thus φ is well defined.

Let $(\alpha_1, \ldots, \alpha_m), (\beta_1, \ldots, \beta_s) \in \mathbb{A}_n$ and suppose that $\varphi(\alpha_1, \ldots, \alpha_m) = \varphi(\beta_1, \ldots, \beta_s)$. By definition, we get that $m = s$ and $\varphi(\alpha_i) = \varphi(\beta_i)$ for all $i \in \{1, 2, \ldots, m\}$. Hence $\alpha_i = \beta_i$ for all $\{1, 2, \ldots, m\}$ and so $(\alpha_1, \ldots, \alpha_m) = (\beta_1, \ldots, \beta_s)$. Thus φ is injective.

It remains to prove that φ is surjective. In order to do so, let $\beta = (\beta_1, \ldots, \beta_\ell) \in \mathbb{F}_{2n}$. Notice that $\beta_1 = 1$ or $\beta_1 = 2$. Suppose first that $\beta_1 = 1$. In this case, since $\beta \in \mathbb{F}_{2n}$, we have that $\beta_i = 1$ for some $i \in \{2, \ldots, \ell\}$. Let $j \in \{2, \ldots, \ell\}$ be the lowest index such that $\beta_j = \mathtt{1}$. If $j = \ell$, then $\beta = \varphi(\ell - 1)$. If $j = 2$, then we get that $\beta = (\varphi(\mathtt{1}_1), \beta')$ for some $\beta' \in \mathbb{F}_{2n-2}$. Now, if $2 < j < \ell$, then $\beta = (\varphi(j-1), \beta')$ for some $\beta' \in \mathbb{F}_{2n-2j+2}$. If, on the contrary, $\beta_1 = \mathtt{2}$, then we have that $\beta = (\varphi(\mathtt{1}_2), \beta')$ for some $\beta' \in \mathbb{F}_{2n-2}$.

We conclude from the previous analysis that $\beta = \varphi(\ell-1)$ or $\beta = (\varphi(\alpha_1), \beta')$ for some $\alpha_1 \in \{\mathtt{1}_1, \mathtt{1}_2, \mathtt{j-1}\}$ and $\beta' \in \mathbb{F}_{2n-2\alpha_1}$. If $\beta = \varphi(\ell-1)$, then we are through. Otherwise, we repeat the argument given above with β replaced by β'. Repeating the above argument, as many times as needed, we finally obtain that $\beta = \varphi(\alpha_1, \ldots, \alpha_m)$ for some $m \geq 2$ and $\alpha_i \in \{\mathtt{1}_1, \mathtt{1}_2, \mathtt{2}, \ldots, \ell - 1\}$ for all $i \in \{1, \ldots, m\}$. Thus φ is surjective, and so the proof of Theorem 4.6 is complete. For example, if $\beta = (\mathtt{2}, \mathtt{1}, \mathtt{2}, \mathtt{1}, \mathtt{1}, \mathtt{1}, \mathtt{1}, \mathtt{2}, \mathtt{2}, \mathtt{1}, \mathtt{2})$, then

$$\beta = (\varphi(\mathtt{1}_2), \varphi(\mathtt{2}), \varphi(\mathtt{1}_1), \varphi(\mathtt{3}), \varphi(\mathtt{1}_2)).$$ □

By using the above theorem and taking into account that the compositions of n use parts at most n, we have the following corollary.

Corollary 4.4. *Let $k \geq 2$ an integer. Then*

$$C_k(n) = |\mathbb{A}_n| = F_{2n+1} \quad \textit{holds for all } n, \quad 1 \leq n \leq k.$$

The following result establishes a relationship between compositions with parts in the set $\{\mathtt{1}_1, \mathtt{1}_2, \mathtt{2}, \ldots, \mathtt{k}\}$ and the $k-$generalized Pell numbers.

Theorem 4.7. *The generalized Pell number $P_{n+1}^{(k)}$ counts the number of compositions of n with parts in the set $\{\mathtt{1}, \mathtt{2}, \ldots, \mathtt{k}\}$ such that the summand $\mathtt{1}$ can take two colors. Namely,*

$$C_k(n) = P_{n+1}^{(k)}, \qquad \textit{for all} \quad n \geq 1. \tag{4.2}$$

Proof. Let σ be a generalized bi–colored composition of n with parts in the set $\{\mathtt{1}, \mathtt{2}, \ldots, \mathtt{k}\}$. If σ starts with $\mathtt{1}$, then it must be followed by a bi–colored generalized composition of $n-1$ with parts in the set $\{\mathtt{1}, \mathtt{2}, \ldots, \mathtt{k}\}$. Since the summand $\mathtt{1}$ can take two colors, we have $2C_k(n-1)$ possibilities for σ in this case. Now, if σ starts with $\sigma_1 \in \{\mathtt{2}, \mathtt{3}, \ldots, \mathtt{k}\}$, then σ must be followed by a composition of $n - \sigma_1$. Thus, by the addition principle, the number of generalized bi–colored compositions of n with parts in the set $\{\mathtt{1}, \mathtt{2}, \ldots, \mathtt{k}\}$ is given by $C_k(n) = 2C_k(n-1) + C_k(n-2) + \cdots + C_k(n-k)$. Finally, note that $C_k(n)$ satisfies the $k-$generalized Pell recurrence with $C_k(1) = 2 = P_2^{(k)}$ and $C_k(2) = 5 = P_3^{(k)}$. This proves (4.2). □

Finally, from Corollary 4.4 we deduce the following statement, which was also proved by Kiliç [81] by using arithmetic arguments.

Corollary 4.5. *Let $k \geq 2$ be an integer. Then*

$$P_{n+1}^{(k)} = F_{2n+1} \quad \textit{holds for all} \quad 1 \leq n \leq k.$$

Chapter 5

Common values of generalized Fibonacci and Pell sequences

In this chapter, we find all coincidences between $\ell-$Fibonacci and $k-$Pell numbers. That is, we find all the solutions of the Diophantine equation $P_n^{(k)} = F_m^{(\ell)}$ in positive integers n, k, m, ℓ with $k, \ell \geq 2$. This work continues and extends prior results which dealt with the above problem for some particular cases of k and ℓ. In particular, it extends the previous work [5] that found all Fibonacci numbers in the Pell sequence.

5.1 Introduction

The *Lucas sequences* $U_n(P, Q)$ and $V_n(P, Q)$ are certain linear recursive integer sequences satisfying the relation

$$x_n = Px_{n-1} - Qx_{n-2},$$

where P and Q are fixed integers. Famous examples of Lucas sequences include the *Fibonacci numbers*, *Lucas numbers*, *Pell numbers*, *Pell–Lucas numbers*, *Mersenne numbers* (see [123, A000225]) and Fermat numbers (see [123, A000215]).

There are many papers in the literature which discuss the intersection problem between linear recurrence sequences. For example, in 2011, Alekseyev [5] established that $F^{(2)} \cap P^{(2)} = \{0, 1, 2, 5\}$ and $L^{(2)} \cap P^{(2)} = \{1, 2, 29\}$ using properties of Lucas sequences, homogeneous quadratic Diophantine equations and Thue equations. For linear recurrence sequences of order k, we mention that Bravo, Gómez and Herrera [24] found all generalized Fibonacci numbers which are Pell numbers, while Bravo and Herrera [29] determined all $k-$Fibonacci and $k-$Lucas numbers which are Fermat numbers. In addition to this, Bravo et al. [23] looked for all $k-$Fibonacci numbers which are Mersenne numbers, whereas Bravo and Herrera [30] found all Fibonacci numbers that are generalized Pell numbers. Recently, Hernane et.al. [70] characterized all Fermat and Mersenne numbers that can be represented as a product of two $k-$Fibonacci numbers, while Normenyo et al. [109] solved some intersection problems similar to those discussed above but involving $k-$Pell numbers. For the intersection of generalized Lucas sequences, we refer the reader to [116].

At this point, it is worth mentioning that Noe and Post [106] in 2013 proposed a conjecture about coincidences between terms of generalized Fibonacci sequences, which was proved by Bravo and Luca [36], and also independently by Marques [91]. A problem similar to the previous one with $k-$Lucas sequences was studied by Bravo et.al. [18].

In a similar vein, in this chapter we investigate the problem of determining

$$\bigcup_{k\geq 2,\,\ell\geq 2} P^{(k)} \cap F^{(\ell)}$$

extending the previous works in [5, 24, 30]. We mention that Mignotte (see [100]) proved that under some technical conditions (for example that the sequences have dominant roots which are multiplicatively independent) only a finite number of coincidences between two fixed linear recurrence sequences can occur. This applies to us (but see the Conjecture in Section 5.1) in the context of showing that $P^{(k)} \cap F^{(\ell)}$ is finite for fixed $k, \ell \geq 2$ but for us k, ℓ are variables as well.

Here, we determine all the solutions of the Diophantine equation

$$P_n^{(k)} = F_m^{(\ell)}, \tag{5.1}$$

in positive integers n, k, m, ℓ with $k, \ell \geq 2$. Our main result is as follows.

Theorem 5.1. *The only solutions (n, k, m, ℓ) of Diophantine equation (5.1) in positive integers $n, k \geq 2, m, \ell \geq 2$ are:*

(*i*) *the parametric family of solutions (n, k, m, ℓ) with $\ell = 2$, namely*

$$(n, k, m, \ell) = (t, k, 2t-1, 2) \quad \textit{for} \quad 1 \leq t \leq k+1;$$

(*ii*) *the sporadic solutions:*

$$\begin{array}{rcll} 1 & = & P_1^{(k)} = F_1^{(\ell)} & \textit{for all} \quad k \geq 2 \quad \textit{and} \quad \ell \geq 3; \\ 1 & = & P_1^{(k)} = F_2^{(\ell)} & \textit{for all} \quad k, \ell \geq 2; \\ 2 & = & P_2^{(k)} = F_3^{(\ell)} & \textit{for all} \quad k \geq 2 \quad \textit{and} \quad \ell \geq 3; \\ 13 & = & P_4^{(k)} = F_6^{(3)} & \textit{for all} \quad k \geq 3; \\ 29 & = & P_5 = F_7^{(4)}. & \end{array}$$

Corollary 5.1. *The only power of 2 in $P^{(k)}$ is $P_2^{(k)} = 2$ for all $k \geq 2$.*

To conclude this section we point out some differences between this work and previous ones. In the Fibonacci and Pell case, namely when $k = \ell = 2$, several well known divisibility properties were used by Alekseyev in [5] to solve the problem. Such divisibility properties for higher order recurrences are not known and not expected to hold and therefore it is necessary to attack the problem differently. Our proof combines linear forms in logarithms, reduction techniques and some arithmetic properties of the sequences $P^{(k)}$ and $F^{(\ell)}$.

Furthermore, equation (5.1) involves two distinct higher order sequences and 4 variables unlike the works in [5] and [24, 30] which involve only 2 or 3 variables, respectively. In [36], Bravo and Luca solved a similar equation with 4 variables, namely $F_n^{(k)} = F_m^{(\ell)}$ but since this equation involved the same sequence, it was possible to use symmetry and assume that $k \geq \ell$. In this paper, our equation connects different sequences therefore k and ℓ are independent. In addition to this, in this last equation in order to make sure that linear forms in logarithms involved are nonzero, the work [36] invoked the fact that the largest roots of the characteristic equations for $F^{(k)}$ and $F^{(\ell)}$ are multiplicatively independent when $k \neq \ell$, a fact easily checked using Mignotte's result on multiplicative relations among conjugates of Pisot numbers. For us, we do not know if the largest roots of $F^{(k)}$ and $P^{(\ell)}$ are muliplicatively independent in general and propose this as a conjecture. However, for our practical purposes we only needed it to hold in a finite range $(k = \ell \leq 800)$, where we checked numerically that it is true. In addition, we needed results and estimates for $P^{(k)}$ similar to those found previously for $F^{(\ell)}$ which have not yet appeared anywhere else in the literature. In particular, we needed to study the degree and the logarithmic height of certain algebraic numbers appearing in the dominant term of the Binet–type formula for $P^{(k)}$. Such results will be useful in order to attack other Diophantine problems involving $P^{(k)}$ such that their largest prime factors, whether they are repdigits, etc.

5.2 Preliminary results

Next, we present three technical lemmas which are keys in the proof of Theorem 5.1. We begin with an estimate of nth $k-$Pell number, which is a direct consequence of Lemma 2.2 from [30].

Lemma 5.1. *Let $k \geq 2$ and suppose that $2n-1 \geq k/2$. If $n < \varphi^{k/2}$, then*

$$P_n^{(k)} = \frac{\varphi^{2n-1}}{\sqrt{5}}(1+\zeta_p) \quad \textit{where} \quad |\zeta_p| < \frac{32}{\varphi^{k/2}}.$$

Now, we need the following technical result to prove Lemma 5.3 and that will be useful in Section 5.4.

Lemma 5.2. *Consider the function $g_k(z)$ defined in (3.10). If η is an algebraic integer of degree d, then $g_k(\eta)$ has degree d as well.*

Proof. Let $\eta_1, \ldots, \eta_d$ be all the conjugates of η. If $g_k(\eta)$ has degree smaller than d, then there exist $i \neq j$ in $\{1, \ldots, d\}$ such that $g_k(\eta_i) = g_k(\eta_j)$. Thus

$$\begin{aligned} 0 &= g_k(\eta_i) - g_k(\eta_j) \\ &= \frac{(\eta_i - \eta_j)((k+1)(\eta_i\eta_j - \eta_i - \eta_j) + 2k + 1)}{((k+1)\eta_i^2 - 3k\eta_i + k - 1)((k+1)\eta_j^2 - 3k\eta_j + k - 1)}, \end{aligned}$$

which implies

$$(k+1)(\eta_i\eta_j - \eta_i - \eta_j + 2) = 1.$$

Since η_i and η_j are algebraic integers, we get that $(k+1) \mid 1$, a contradiction. □

We end this section with the following upper bound for $h(g_k(\gamma))$ that will also be useful in Section 5.4.

Lemma 5.3. *For $k \geq 2$, we have that $h(g_k(\gamma)) < 4\log\varphi + \log(k+1)$.*

Proof. We first note that, by Lemma 5.2, $g_k(\gamma)$ has degree k. Let $\mathbb{L} := \mathbb{Q}(\gamma)$ and let a_k be the leading coefficient of the minimal primitive polynomial of $g_k(\gamma)$ over $\mathbb{Z}$. Put

$$H_k(x) = \prod_{i=1}^{k}(x - g_k(\gamma_i)) \in \mathbb{Q}[x] \quad \text{and} \quad \mathcal{N} = \mathbb{N}_{\mathbb{L}/\mathbb{Q}}((k+1)\gamma^2 - 3k\gamma + k - 1) \in \mathbb{Z}.$$

Note that $\mathcal{N}H_k(x) \in \mathbb{Z}[x]$ vanishes at $g_k(\gamma)$ and so a_k divides $|\mathcal{N}|$. But

$$|\mathcal{N}| = \left|\prod_{i=1}^{k}((k+1)\gamma_i^2 - 3k\gamma_i + k - 1)\right| = (k+1)^k \left|\prod_{i=1}^{k}(c_k - \gamma_i)(d_k - \gamma_i)\right|,$$

where

$$(c_k, d_k) := \left(\frac{3k + \sqrt{5k^2+4}}{2(k+1)}, \frac{3k - \sqrt{5k^2+4}}{2(k+1)}\right) \tag{5.2}$$

are the roots of $(k+1)z^2 - 3kz + k - 1$. Since

$$|\Phi_k(y)| < \max\{y^k, 1 + y + \cdots + y^{k-2} + 2y^{k-1}\} < \varphi^{2k} \quad \text{for all} \quad 0 < y < \varphi^2,$$

and $0 < d_k < c_k < \varphi^2$, which are easily seen, it follows that $a_k < \varphi^{4k}(k+1)^k$. By using this and Lemma 3.2(e), we obtain

$$h(g_k(\gamma)) = \frac{1}{k}\left(\log a_k + \sum_{i=1}^{d} \log\max\{|g_k(\gamma_i)|, 1\}\right) < 4\log\varphi + \log(k+1). \qquad \square$$

5.3 A relation between n and m

Assume from now on that (n, k, m, ℓ) is a solution of equation (5.1). Suppose further that $\min\{k, \ell\} \geq 3$ since the cases $k = 2$ and $\ell = 2$ were already solved in [24] and [30], respectively. With $\ell = 2$ we obtain the parametric family of solutions $(n, k, m, \ell) = (t, k, 2t-1, 2)$ for $1 \leq t \leq k+1$ and the solutions $(n, k, m, \ell) = (1, k, 2, 2)$ for all $k \geq 2$, while with $k = 2$ we get the solutions $(n, k, m, \ell) = (1, 2, 1, \ell), (1, 2, 2, \ell), (2, 2, 3, \ell)$ for all $\ell \geq 2$ and the solution $(n, k, m, \ell) = (5, 2, 7, 4)$. We may also assume that $m \geq 3$ and $n \geq 2$ since $F_1^{(\ell)} = F_2^{(\ell)} = P_1^{(k)} = 1$ for all $k \geq 2$, $\ell \geq 2$.

Let us now suppose that $3 \leq n \leq k+1$. Then, by (3.3), equation (5.1) is transformed into the equation $F_{2n-1} = F_m^{(\ell)}$ to be resolved in positive integers n, m, ℓ with $\ell \geq 3$. But this last equation was studied by Bravo and Luca in [36]. By the main result of [36], we have that the only possibilities are $F_1 = F_1^{(\ell)}$, $F_1 = F_2^{(\ell)}$, $F_3 = F_3^{(\ell)}$ for all $\ell \geq 3$ and $F_7 = F_6^{(3)}$. These give the solutions $(n, k, m, \ell) = (1, k, 1, \ell), (1, k, 2, \ell), (2, k, 3, \ell)$ for all $k, \ell \geq 3$ and $(n, k, m, \ell) = (4, k, 6, 3)$ for all $k \geq 3$. Thus, from now on we assume that $n \geq k+2$. A quick calculation reveals that equation (5.1) has no solution in the range $n \in [k+2, 7]$. So, $n \geq 8$.

Let us now get a relation between n and m. Indeed, it follows from the exponential growth of $F^{(\ell)}$ and $P^{(k)}$ (see (2.19) and Theorem 3.3 (b)) that

$$\gamma^{n-2} \leq P_n^{(k)} = F_m^{(\ell)} \leq 2^{m-2} \quad \text{and} \quad \alpha^{m-2} \leq F_m^{(\ell)} = P_n^{(k)} \leq \gamma^{n-1}.$$

From the above and using the fact that $\gamma < \varphi^2$, we get $1.27n - 0.55 < m \leq 2n - 1$. Thus, the inequalities

$$1.2n < m \leq 2n - 1 \quad \text{hold for all } n \geq 8. \tag{5.3}$$

5.4 An inequality for n and m in terms of $\max\{k, \ell\}$

For the remainder of this chapter, let us denote $\Gamma := \max\{k, \ell\}$ and $\lambda := \min\{k, \ell\}$. From (2.16) and Theorem 3.3 (a) we get that

$$\left|g_k(\gamma)\gamma^n - f_\ell(\alpha)\alpha^{m-1}\right| = \left|(g_k(\gamma)\gamma^n - P_n^{(k)}) + (F_m^{(\ell)} - f_\ell(\alpha)\alpha^{m-1})\right| < 1.$$

Dividing both sides of the above inequality by $f_\ell(\alpha)\alpha^{m-1}$, we conclude that

$$\left|\gamma^n \alpha^{-(m-1)} g_k(\gamma)(f_\ell(\alpha))^{-1} - 1\right| < \frac{4}{\alpha^m}. \tag{5.4}$$

In order to apply Matveev's result to the left–hand side of (5.4), we take $t := 3$,

$$(\eta_1, b_1) := (\gamma, n), \quad (\eta_2, b_2) := (\alpha, -(m-1)) \quad \text{and} \quad (\eta_3, b_3) := (g_k(\gamma)(f_\ell(\alpha))^{-1}, 1).$$

We note that η_1, η_2, η_3 are positive real numbers and belong to $\mathbb{K} := \mathbb{Q}(\alpha, \gamma)$. So, we can take $D := \Gamma^2$ because $[\mathbb{K} : \mathbb{Q}] \leq k\ell \leq \Gamma^2$. Since $h(\eta_1) = (\log \gamma)/k$ and $h(\eta_2) = (\log \alpha)/\ell$, we choose $A_1 := 2\Gamma \log \varphi$ and $A_2 := \Gamma \log 2$. By Lemma 5.3, we get that

$$h(\eta_3) \leq h(f_\ell(\alpha)) + h(g_k(\gamma)) < 2\log \ell + 4\log \varphi + \log(k+1) \leq 6\log \Gamma,$$

where in the last inequality we used the facts that $h(f_\ell(\alpha)) < 2\log \ell$ for all $\ell \geq 2$ (by (2.29)) and $4\log \varphi + \log(k+1) \leq 4\log k$ for all $k \geq 3$. So, we can take $A_3 := 6\Gamma^2 \log \Gamma$. Furthermore, by (5.3), we take $B := m$.

We now need to show that the left–hand side of (5.4) is not zero. Indeed, if this were zero, we would then get that $g_k(\gamma)\gamma^n = f_\ell(\alpha)\alpha^{m-1}$. First, we show that $f_\ell(\alpha)\alpha^{m-1}$ has degree ℓ. Indeed, if not, then like in the proof of Lemma 5.2 there exist $i, j \in \{1, \ldots, \ell\}$,

$i < j$, such that $f_\ell(\alpha_i)\alpha_i^{m-1} = f_\ell(\alpha_j)\alpha_j^{m-1}$. By conjugating with a Galois automorphism which sends α_i to $\alpha = \alpha(1)$, we may assume that $i = 1$. Thus

$$\varphi^2 \leq \varphi^{m-1} \leq \left(\frac{\alpha}{|\alpha_j|}\right)^{m-1} = \frac{|f_\ell(\alpha_j)|}{f_\ell(\alpha)} < 2,$$

a contradiction. A similar argument yields that $g_k(\gamma)\gamma^n$ has degree k. Since in fact $\mathbb{Q}(f_\ell(\alpha)\alpha^{m-1}) \subseteq \mathbb{Q}(\alpha)$ and have the same degree ℓ, these two fields are equal and similarly since $\mathbb{Q}(g_k(\gamma)\gamma^n) \subseteq \mathbb{Q}(\gamma)$ and have the same degree k, these last two fields are also equal. Consequently, we deduce that $\mathbb{Q}(\alpha) = \mathbb{Q}(\gamma)$ and so $k = \ell$. Computing norms and taking reciprocals we get

$$|N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(f_k(\alpha))|^{-1} = |N_{\mathbb{Q}(\gamma)/\mathbb{Q}}(g_k(\gamma))|^{-1}.$$

The left–hand side was estimated in (2.15). It is

$$\frac{2^{k+1}k^k - (k+1)^{k+1}}{(k-1)^2} < \frac{2^{k+1}k^k}{(k-1)^2}.$$

We need to show that the right–hand side is bigger. The right–hand side is

$$\prod_{i=1}^{k} \left| \frac{(k+1)\gamma_i^2 - 3k\gamma_i + k - 1}{\gamma_i - 1} \right|.$$

The denominators is

$$\left| \prod_{i=1}^{k} (\gamma_i - 1) \right| = |\Phi_k(1)| = k.$$

For the numerator, we write it as:

$$(k+1)^k \prod_{i=1}^{k} |(\gamma_i - c_k)(\gamma_i - d_k)| = (k+1)^k \frac{|c_k^{k-1}(c_k^2 - 3c_k + 1) + 1||d_k^{k-1}(d_k^2 - 3d_k + 1) + 1|}{|(c_k - 1)(d_k - 1)|},$$

where (c_k, d_k) is given by (5.2). In proof of Lemma 3.1, note that $c_k + 1/k$ is an increasing function of k which implies that c_k is increasing as well. So, we have $c_k \geq c_5 > 2.1$ for $k \geq 5$. Since the product of c_k and d_k is $(k-1)/(k+1) < 1$, we get that $d_k < 1/2.1$. Thus,

$$|d_k^{k-1}(d_k^2 - 3d_k + 1) + 1| > 1 - \frac{6}{2.1^k} > 0.8$$

for $k \geq 5$. Further,

$$(k+1)c_k^2 - 3(k+1)c_k + (k+1) = -3c_k + 2 < -4.3.$$

So $|c_k^2 - 3c_k + 1| > 4.3/(k+1)$. Hence,

$$|c_k^{k-1}(c_k^2 - 3c_k + 1) + 1| > \frac{2.1^{k-1} \times 4.3}{k+1} - 1 > \frac{2^{k+1}}{k+1}.$$

Finally, by the Viete relations from the polynomial $(k+1)z^2 - 3kz + 1$ for having c_k, d_k as roots, we have

$$|(c_k - 1)(d_k - 1)| = |c_k d_k - (c_k + d_k) + 1| = \left|\frac{k-1}{k+1} - \frac{3k}{k+1} + 1\right| = \frac{k}{k+1}.$$

Thus,

$$|N_{\mathbb{L}/\mathbb{Q}}(g_k(\gamma))|^{-1} > \frac{(k+1)^{k+1}}{k^2} \cdot \frac{2^{k+1} \times 0.8}{k+1} = 2^{k+1} \times 0.8\ \frac{(k+1)^k}{k^2}.$$

Then, it suffices that

$$2^{k+1} \times 0.8\ \frac{(k+1)^k}{k^2} > \frac{2^{k+1} k^k}{(k-1)^2},$$

or equivalently

$$\left(1 + \frac{1}{k}\right)^k > \frac{5}{4}\left(\frac{k}{k-1}\right)^2.$$

The left–hand side is increasing with a limit of e and the right–hand side is decreasing with a limit of $5/4$. Thus, the desired inequality is satisfied for $k \geq 5$. For $k = 3, 4$ one computes the corresponding norms. Hence, the left–hand side of (5.4) is not zero.

Applying Matveev's theorem to the inequality (5.4), we obtain

$$-\log\left|\gamma^n \alpha^{-(m-1)} g_k(\gamma)(f_\ell(\alpha))^{-1} - 1\right| < 3.45 \times 10^{12}\ \Gamma^8 \log^2 \Gamma \log m, \tag{5.5}$$

where we used that the inequalities $1 + \log \Gamma^2 \leq 3 \log \Gamma$ and $1 + \log m \leq 2 \log m$ hold for all $\Gamma, m \geq 3$. Taking logarithms on both sides of (5.4) and comparing the resulting inequality with (5.5) we get, after performing the respective calculations, that

$$\frac{m}{\log m} < 7 \times 10^{12}\ \Gamma^8 \log^2 \Gamma. \tag{5.6}$$

In order to find an upper bound on m in terms of Γ and $\log \Gamma$, we use the fact that the inequality $x/\log x < T$ implies $x < 2T \log T$ whenever $T > 4$ (see Lemma 2.9). Putting $x := m$ and $T := 7 \times 10^{12}\ \Gamma^8 \log^2 \Gamma$, inequality (5.6) yields $m < 5.1 \times 10^{14}\ \Gamma^8 \log^3 \Gamma$. In the above we used that $\log T \leq 36 \log \Gamma$ holds for all $\Gamma \geq 3$. In summary, we have proved the following intermediate result.

Lemma 5.4. *If (n, k, m, ℓ) is a solution of equation (5.1) with $\lambda \geq 3$ and $n \geq k + 2$, then $n \geq 8$, $m \geq 10$ and*

$$1.2n < m < 5.1 \times 10^{14}\ \Gamma^8 \log^3 \Gamma.$$

In particular, $n < m < \Gamma^{14}$ for all $\Gamma > 800$.

5.5 An expression involving λ and Γ

Let us mention two important consequences of Lemma 5.4. We begin with the next result, which is a key point for finding and reducing bounds for large values of Γ.

Lemma 5.5. *Let (n, k, m, ℓ) be a solution of equation (5.1) with $\lambda \geq 3$ and $n \geq k+2$. Suppose that $n < \varphi^{k/2}$ and $m < 2^{\ell/2}$. Then*

(a) $\lambda < 1.4 \times 10^{13} \log n$. *Moreover, if $\Gamma > 800$, then $\lambda < 2 \times 10^{14} \log \Gamma$.*

(b) *If $\lambda \geq 18$, then*

$$0 < \left| (m-2)\frac{\log 2}{\log \varphi} - (2n-1) + \frac{\log \sqrt{5}}{\log \varphi} \right| < \frac{138}{\varphi^{\lambda/2}}. \tag{5.7}$$

Proof. We begin by observing that Lemmas 2.4 and 5.1 together with equation (5.1) imply that

$$\left| \frac{\varphi^{2n-1}}{\sqrt{5}} - 2^{m-2} \right| < \frac{32}{\varphi^{k/2}} \cdot \frac{\varphi^{2n-1}}{\sqrt{5}} + \frac{2^{m-2}}{2^{\ell/2}}. \tag{5.8}$$

We need to distinguish two cases. Suppose first that $\varphi^{2n-1}/\sqrt{5} \geq 2^{m-2}$. Dividing both sides of (5.8) by $\varphi^{2n-1}/\sqrt{5}$, we get

$$\left| 2^{m-2}\varphi^{-(2n-1)}\sqrt{5} - 1 \right| < \frac{32}{\varphi^{k/2}} + \frac{1}{2^{\ell/2}} < \frac{33}{\varphi^{\lambda/2}}.$$

Now, if $\varphi^{2n-1}/\sqrt{5} < 2^{m-2}$, then we can divide (5.8) by 2^{m-2} to obtain

$$\left| 2^{-(m-2)}\varphi^{2n-1}(\sqrt{5})^{-1} - 1 \right| < \frac{32}{\varphi^{k/2}} + \frac{1}{2^{\ell/2}} < \frac{33}{\varphi^{\lambda/2}}.$$

In any case we have an expression like

$$\left| 2^{\varepsilon(m-2)}\varphi^{-\varepsilon(2n-1)}(\sqrt{5})^{\varepsilon} - 1 \right| < \frac{33}{\varphi^{\lambda/2}}, \quad \text{where} \quad \varepsilon \in \{\pm 1\}. \tag{5.9}$$

We next apply Matveev's result to the left–hand side of (5.9) with $t := 3$ and the parameters

$$(\eta_1, b_1) := (2, \varepsilon(m-2)), \quad (\eta_2, b_2) := (\varphi, -\varepsilon(2n-1)) \quad \text{and} \quad (\eta_3, b_3) := (\sqrt{5}, \varepsilon).$$

Note that the algebraic number field $\mathbb{K} := \mathbb{Q}(\varphi)$ contains η_1, η_2, η_3 and has degree $D := 2$. Moreover, the left–hand side of (5.9) is not zero. Indeed, if this were zero, we would then get that $2^{m-2} = \varphi^{2n-1}/\sqrt{5}$ and so $\varphi^{4n-2} \in \mathbb{Q}$, which is not possible.

Since $h(\eta_1) = \log 2$, $h(\eta_2) = (\log\varphi)/2$ and $h(\eta_3) = (\log 5)/2$, we can take $A_1 := 2\log 2$, $A_2 := \log\varphi$ and $A_3 := \log 5$. By (5.3), we can take $B := 2n$. Applying Matveev's theorem to inequality (5.9), we deduce that

$$-\log\left|2^{\varepsilon(m-2)}\varphi^{-\varepsilon(2n-1)}(\sqrt{5})^{\varepsilon} - 1\right| < 3.2\times 10^{12}\ \log n, \tag{5.10}$$

where we used that $1 + \log(2n) \leq 3\log n$ holds for all $n \geq 3$. By comparing (5.9) with (5.10) we get $\lambda < 1.4\times 10^{13}\log n$. For the second part of (a), since $\Gamma > 800$, it follows from Lemma 5.4 that $\log n < 14\log\Gamma$ and hence $\lambda < 2\times 10^{14}\log\Gamma$. This proves (a). To prove (b), we put

$$z := \varepsilon\left((m-2)\log 2 - (2n-1)\log\varphi + \log(\sqrt{5})\right),$$

and observe that (5.9) can be written as $|e^z - 1| < 33/\varphi^{\lambda/2}$. Note that $z \neq 0$. If $z > 0$, then we can apply Lemma 2.10 (a) to obtain $|z| < 33/\varphi^{\lambda/2}$. If, on the contrary, $z < 0$, then $|e^z - 1| < 1/2$ because $\lambda \geq 18$. Thus, by Lemma 2.10 (b), we have $|z| < 2|e^z - 1| < 66/\varphi^{\lambda/2}$. In any case, we get that $|z| < 66/\varphi^{\lambda/2}$ holds for all $\lambda \geq 18$. Replacing z in the above inequality by its formula and dividing it across by $\log\varphi$, we get that the inequalities

$$0 < \left|(m-2)\frac{\log 2}{\log\varphi} - (2n-1) + \frac{\log(\sqrt{5})}{\log\varphi}\right| < \frac{138}{\varphi^{\lambda/2}} \quad \text{hold for all} \quad \lambda \geq 18,$$

where we used that $66/\log\varphi < 138$. This completes the proof of the lemma. □

By applying the above lemma, we derive the following result.

Lemma 5.6. *The inequalities* $\lambda < 2\times 10^{14}\log\Gamma$ *and* $\log\lambda < 7\log\Gamma$ *hold for all* $\Gamma > 800$.

Proof. Suppose $\Gamma > 800$. We consider the two possible cases for Γ. If $\Gamma = k$, then $n < \varphi^{k/2}$ and $m < k^{14}$ by Lemma 5.4. Note that either $2^{\ell/2} \leq m$ or $m < 2^{\ell/2}$. If $2^{\ell/2} \leq m$, then $\ell < (2/\log 2)(14\log k) < 41\log k$; i.e., $\lambda < 41\log\Gamma$. If, on the contrary, $m < 2^{\ell/2}$, then $\lambda < 2\times 10^{14}\log\Gamma$ by Lemma 5.5 (a). We now assume that $\Gamma = \ell$. Here, by Lemma 5.4 once more, we get that $m < 2^{\ell/2}$ and $n < \ell^{14}$. Similarly, if $\varphi^{k/2} \leq n$, then $k < 59\log\ell$, while if $n < \varphi^{k/2}$, then, by Lemma 5.5 (a), we obtain $\lambda < 2\times 10^{14}\log\Gamma$. Thus, in any case we have that $\lambda < 2\times 10^{14}\log\Gamma$, which in it turns implies that $\log\lambda < 7\log\Gamma$ because $\Gamma > 800$. This finishes the proof. □

5.6 The case of small Γ

In this section, we treat the cases when $\Gamma \in [3, 800]$. We shall use several times Lemma 2.7 in order to lower the upper bounds for our variables. For this, we write

$$z := n \log \gamma - (m-1) \log \alpha + \log \left(g_k(\gamma)(f_\ell(\alpha))^{-1} \right).$$

Therefore, inequality (5.4) can be rewritten as $|e^z - 1| < 4/\alpha^m$. In this case, $z \neq 0$. If $z > 0$, then we can apply Lemma 2.10 (a) to obtain $|z| < 4/\alpha^m$. Now, if $z < 0$, then $|e^z - 1| < 1/2$ since $m \geq 10$. Thus, by Lemma 2.10 (b), we have that $|z| < 2|e^z - 1| < 8/\alpha^m$. In any case, we get that $|z| < 8/\alpha^m$. Replacing z in the above inequality by its formula and dividing it across by $\log \alpha$, we get that

$$0 < |n\tau - (m-1) + \mu| < AB^{-m}, \tag{5.11}$$

where

$$\tau := \tau(k, \ell) = \frac{\log \gamma}{\log \alpha}, \quad \mu := \mu(k, \ell) = \frac{\log \left(g_k(\gamma)(f_\ell(\alpha))^{-1} \right)}{\log \alpha}, \quad A := 16 \quad \text{and} \quad B := \alpha.$$

We would like to know that τ is an irrational number. Unfortunately, we do not know that in general. So, we propose it as a conjecture:

Conjecture 5.1. *Show that $\alpha = \alpha(\ell)$ and $\gamma = \gamma(k)$ are multiplicatively independent for all $k \geq 2,\ \ell \geq 2$ except when $k = \ell = 2$.*

But we can prove it for the instances we need. Namely, assume that $\tau = a/b$ for some coprime positive integers a and b. Thus, $\alpha^a = \gamma^b$. Using a similar argument to that used for proving that the left-hand side of (5.4) is not zero, one shows easily that α^a and γ^b have degrees ℓ and k, respectively. Hence, $k = \ell$ and therefore $\mathbb{Q}(\alpha) = \mathbb{Q}(\gamma)$. Conjugating by Galois automorphisms and taking absolute values, we get that for each $i \in \{1, \ldots, \ell\}$ there exists $j \in \{1, \ldots, k\}$ such that $|\alpha_i|^a = |\gamma_j|^b$. Since α and γ are Pisot numbers, it follows from a result of Mignotte [101], that $|\alpha_u| = |\alpha_v|$ if and only if α_u and α_v are complex conjugates and a similar result holds for the γ's. Since Ψ_k has exactly one real root when k is odd and two real roots when k is even and the rest are complex conjugate, it follows that the set $\{|\alpha_i| : i = 1, \ldots, k\}$ contains exactly $\lfloor k/2 \rfloor + 1$ distinct elements and the same is true for the set $\{|\gamma_j| : j = 1, \ldots, k\}$. Let α_2, γ_2 be roots different than α and γ, respectively, of largest absolute values among the roots of Ψ_k and Φ_k, respectively. Then the relation $\alpha^a = \gamma^b$ yields by conjugation $|\alpha_2|^a = |\gamma_2|^b$. This gives a procedure to check that in fact τ is irrational. Namely, for each $k \in [4, 800]$, we compute $|\alpha_2|$ and $|\gamma_2|$. If it were true that $\alpha^a = \gamma^b$, then also $|\alpha_2|^a = |\gamma_2|^b$, so

$$\frac{\log \gamma}{\log \alpha} = \frac{\log |\gamma_2|}{\log |\alpha_2|} \quad \left(= \frac{a}{b} \right).$$

With the computer, we computed both sides of the above equality for all $k \in [4, 800]$ and checked that they are in fact different. The above procedure fails when $k = 3$ because $|\alpha_2| = \alpha^{-1/2}$, $|\gamma_2| = \gamma^{-1/2}$ so $(\log \gamma)/(\log \alpha) = (\log |\gamma_2|)/(\log |\alpha_2|)$. In this case we compute the discriminants D_Ψ and D_Φ of the polynomials $\Psi_3(x)$ and $\Phi_3(x)$, respectively, obtaining $D_\Psi = -44$ and $D_\Phi = -87$. If it were true that $\mathbb{Q}(\alpha) = \mathbb{Q}(\gamma)(= \mathbb{K})$, then we would have $D_\Psi = \Delta_\mathbb{K} I(\alpha)^2$ and $D_\Phi = \Delta_\mathbb{K} I(\gamma)^2$, where $\Delta_\mathbb{K}$ is the discriminant of $\mathbb{K}$ and $I(\alpha)$, $I(\gamma)$ are the indices of $\mathbb{Z}[\alpha]$, $\mathbb{Z}[\gamma]$ in $\mathcal{O}_\mathbb{K}$, respectively. Thus, we should have that $I(\gamma) = 1$ and so $D_\Psi/D_\Phi = I(\alpha)^2$ is a square of an integer, which is not the case. Thus, α, γ are multiplicatively independent when $k = \ell = 3$ as well. Hence, τ is irrational in our range of Γ.

In order to reduce our upper bound for m, we take $M_\Gamma := \lfloor 5.1 \times 10^{14}\, \Gamma^8 \log^3 \Gamma \rfloor$ (upper bound on n from Lemma 5.4) and we apply Lemma 2.7 to the inequality (5.11) for each $k \in [3, 800]$ and $\ell \in [3, 800]$. A computer search with *Mathematica* revealed that $m \leq 200$. Then, $k + 2 \leq n \leq 200$ implies $k \leq 200$.

Finally, we used *Mathematica* to conclude that equation (5.1) has no solutions in the range $3 \leq k \leq 200$, $3 \leq \ell \leq 800$, $k + 2 \leq n \leq 200$ and $10 \leq m \leq 200$. This completes the analysis when $\Gamma \in [3, 800]$.

5.7 The case of large Γ

From now on, we assume that $\Gamma > 800$. Here, it follows from Lemma 5.4 that

$$n < m < \varphi^{\Gamma/2} < 2^{\Gamma/2} \quad \text{and} \quad n < m < \Gamma^{14}.$$

In this section, our goal is to find absolute upper bounds for the variables of equation (5.1). To do so, we shall distinguish the two possible cases for Γ.

5.7.1 Case $\Gamma = k$

In this case we have that $n < \varphi^{k/2}$. Combining (2.16) and Lemma 5.1 we can conclude that

$$\left| \frac{\varphi^{2n-1}}{\sqrt{5}} - f_\ell(\alpha)\alpha^{m-1} \right| < \frac{32}{\varphi^{k/2}} \cdot \frac{\varphi^{2n-1}}{\sqrt{5}} + \frac{1}{2}.$$

Multiplying both sides of the above inequality by $\sqrt{5}\varphi^{-(2n-1)}$ and taking into account that $\varphi^{2n-1} > \varphi^{k/2}$, we deduce that

$$\left|\alpha^{m-1}\varphi^{-(2n-1)}\sqrt{5}f_\ell(\alpha) - 1\right| < \frac{34}{\varphi^{k/2}}. \tag{5.12}$$

We now want to apply Matveev's theorem to the left–hand side of (5.12). For it, we take $t := 3$ and the parameters

$$(\eta_1, b_1) := (\alpha, m-1), \quad (\eta_2, b_2) := (\varphi, -(2n-1)) \quad \text{and} \quad (\eta_3, b_3) := (\sqrt{5}f_\ell(\alpha), 1).$$

The number field containing η_1, η_2, η_3 is $\mathbb{K} := \mathbb{Q}(\sqrt{5}, \alpha)$ of degree $D \le 2\ell$. As we saw before, $h(\eta_1) = (\log\alpha)/\ell$, $h(\eta_2) = (\log\varphi)/2$ and $h(\eta_3) \le \log\sqrt{5} + \log h(f_\ell(\alpha)) < 3\log\ell$ for all $\ell \ge 3$ by (2.29). Consequently, we can take $A_1 := 2\log 2$, $A_2 := \ell\log\varphi$ and $A_3 := 6\ell\log\ell$. Since $n < 2m$ (see (5.3)), we take $B := 2n$. We need to check that the left–hand side of (5.12) is not zero. Assuming it is, we get $(\sqrt{5}f_\ell(\alpha))^2 = \varphi^{2(2n-1)} \cdot \alpha^{-2(m-1)} \in \mathcal{O}_{\mathbb{K}}$. Let us show that this is false. By (2.13), we have

$$5f_\ell(\alpha)^2 < \frac{45}{16}.$$

Further, for $2 \le i \le \ell$, we have

$$|f_\ell(\alpha_i)| = \frac{|\alpha_i - 1|}{|2 + (\ell+1)(\alpha_i - 2)|} < \frac{2}{(\ell+1) - 2} = \frac{2}{\ell - 1},$$

so that

$$5|f_\ell(\alpha_i)^2| < \frac{20}{(\ell-1)^2}.$$

For $\ell \ge 6$, the right–hand side is $\le 20/25 = 4/5$. Hence, for such ℓ,

$$|N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(5(f_\ell(\alpha)^2)| \le \frac{45}{16}\left(\frac{4}{5}\right)^{\ell-1} \le \frac{45}{16}\left(\frac{4}{5}\right)^5 < 1,$$

so $5f_\ell(\alpha)^2$ cannot be an algebraic integer. One checks with *Mathematica* that $5f_\ell(\alpha)^2$ is not an algebraic integer for $\ell \in \{3, 4, 5\}$ either. Hence, the left–hand side of (5.12) is nonzero.

It then follows from Matveev's theorem that

$$-\log\left|\alpha^{m-1}\varphi^{-(2n-1)}\sqrt{5}f_\ell(\alpha) - 1\right| < 2.1 \times 10^{13}\ \ell^4 \log^2\ell \log n. \tag{5.13}$$

By comparing (5.12) with (5.13) we get $k < 8.8 \times 10^{13}\ell^4\log^2\ell\log n$. But $\log n < 14\log k$ by Lemma 5.4. Hence, $k < 1.3 \times 10^{15}\ell^4\log^2\ell\log k$. We next apply Lemma 5.6 to obtain that

$$k < (1.3 \times 10^{15})(2 \times 10^{14}\log k)^4(7\log k)^2\log k < 1.1 \times 10^{74}\log^7 k,$$

which implies $k < 2 \times 10^{90}$. Thus, $n < m < 1.2 \times 10^{744}$ by Lemma 5.4.

If $m \leq 2^{\ell/2}$ and $\ell \geq 18$, then we apply Lemma 2.7 to inequality (5.7) with the parameters

$$\tau := \frac{\log 2}{\log \varphi}, \quad \mu := \frac{\log \sqrt{5}}{\log \varphi}, \quad A := 138 \quad \text{and} \quad B := \varphi^{1/2}.$$

Clearly τ is an irrational number. We put $M := 1.2 \times 10^{744}$ which is an upper bound on $m - 2$. Then, from Lemma 2.7 we obtain that $\ell < \log(Aq/\varepsilon)/\log B$, where $q \geq 6M$ is a denominator of a convergent of the continued fraction of τ with $\varepsilon = ||\mu q|| - M||\tau q|| > 0$. A computer search with *Mathematica* revealed that $\log(Aq/\varepsilon)/\log B$ is ≤ 7160. Thus, $\ell \leq 7160$. Note that, if $2^{\ell/2} \leq m < 1.2 \times 10^{744}$, then we get $\ell \leq 4950$. We can then conclude from the above analysis that $\ell \in [3, 7160]$.

We still need to lower some upper bounds for our variables. For this, we now put

$$z := (m-1)\log\alpha - (2n-1)\log\varphi + \log(\sqrt{5} f_\ell(\alpha)).$$

This allows us to rewrite inequality (5.12) as $|e^z - 1| < 34/\varphi^{k/2}$. Note that $z \neq 0$ and $|e^z - 1| < 1/2$ since $k > 800$. Hence, Lemma 2.10 gives $|z| < 68/\varphi^{k/2}$. Replacing z by its formula and dividing it across by $\log\varphi$, we get that

$$0 < |(m-1)\tau - (2n-1) + \mu| < AB^{-k}, \tag{5.14}$$

where

$$\tau := \tau(\ell) = \frac{\log\alpha}{\log\varphi}, \quad \mu := \mu(\ell) = \frac{\log(\sqrt{5} f_\ell(\alpha))}{\log\varphi}, \quad A := 142 \quad \text{and} \quad B := \varphi^{1/2}.$$

We next apply Lemma 2.7 to inequality (5.14). For this purpose, we need to show that τ is an irrational number. Indeed, if it were not, then $\tau = a/b$ with coprime positive integers a and b, and so $\alpha^b = \varphi^a$. The number in the right has only two conjugates namely φ^a and $(-\varphi^{-1})^a$, whereas the number in the left has $\ell \geq 3$ conjugates, a contradiction. So, τ is irrational.

Taking $M_\ell := 1.2 \times 10^{744}$ (which is an upper bound on $m - 1$), we apply Lemma 2.7 to the inequality (5.14) for all $\ell \in [3, 7160]$. Using *Mathematica* we found that the maximum value of $\log(Aq/\varepsilon)/\log B$ is ≤ 7200. Hence, $k \leq 7200$ and Lemma 5.4 gives $m < 2.6 \times 10^{48}$. With this new upper bound for $m - 1$ we repeated the process. That is we apply again Lemma 2.7 to the inequality (5.7) with $M := 2.6 \times 10^{48}$ and we obtain that $\ell \in [3, 500]$. With this new range for ℓ and the new upper bound for m, we apply again Lemma 2.7 to the inequality (5.14) to finally obtain $k \leq 530$. But this contradicts our assumption that $\Gamma = k > 800$.

5.7.2 Case $\Gamma = \ell$

In this case we have that $m < 2^{\ell/2}$. Similarly as before, from Lemma 2.4 and Theorem 3.3 (a), we obtain the inequality

$$\left|\gamma^n 2^{-(m-2)} g_k(\gamma) - 1\right| < \frac{2}{2^m} + \frac{1}{2^{\ell/2}} < \frac{3}{2^\theta}, \tag{5.15}$$

where $\theta := \min\{m, \ell/2\}$. We now apply Matveev's theorem once again with the parameters $t := 3$,

$$(\eta_1, b_1) := (\gamma, n), \quad (\eta_2, b_2) := (2, -(m-2)) \quad \text{and} \quad (\eta_3, b_3) := (g_k(\gamma), 1).$$

Here, we take $\mathbb{K} := \mathbb{Q}(\gamma)$ which has degree $D := k$. In this application of Matveev's theorem we take $B := m$, $A_1 := 2\log\varphi$, $A_2 := k\log 2$ and $A_3 := 4k\log k$. Note that if the left–hand side of (5.15) were zero, then we would get that $g_k(\gamma) = 2^{m-2}\gamma^{-n}$. But this implies that $g_k(\gamma)$ is an algebraic integer which contradicts Lemma 3.2 (f). Then, Theorem 2.9 and inequality (5.15) yield, after doing some algebraic calculations, the following upper bound for θ:

$$\theta < 2.3 \times 10^{12}\ k^4 \log^2 k \log m. \tag{5.16}$$

We now write

$$z := n\log\gamma - (m-2)\log 2 + \log(g_k(\gamma)).$$

Thus, from inequality (5.15) we conclude that $|e^z - 1| < 3/2^\theta$. Note that $|e^z - 1| < 1/2$ since $m \geq 10$ and $\ell > 800$. Hence, by Lemma 2.10, we get that $|e^z - 1| < 6/2^\theta$. Dividing the above inequality by $\log 2$, we obtain

$$0 < |n\tau - (m-2) + \mu| < AB^{-\theta}, \tag{5.17}$$

where now

$$\tau := \tau(k) = \frac{\log\gamma}{\log 2}, \quad \mu := \mu(k) = \frac{\log(g_k(\gamma))}{\log 2}, \quad A := 9 \quad \text{and} \quad B := 2.$$

We need to consider the following two subcases.

5.7.2.1 The case $\theta = \ell/2$

Taking into account that $\log m < 14\log\ell$, it follows from inequality (5.16) that $\ell < 6.5 \times 10^{13}\ k^4 \log^2 k \log\ell$. By using this and Lemma 5.6 we have that $\ell < 5.1 \times 10^{72} \log^7 \ell$. Hence, $\ell < 8 \times 10^{88}$ and so $n < m < 7.4 \times 10^{732}$ by Lemma 5.4.

If $\varphi^{k/2} \leq n$, then it is easy to check that $k \leq 7015$. It remains to analyse the case when $n < \varphi^{k/2}$. In this last case we apply Lemma 2.7 to the inequality (5.7) with $M := 7.4 \times 10^{732}$ (which is a upper bound for $m-2$). Here, we get $k \leq 7100$. In any case, we have that $k \in [3, 7100]$.

Taking $M := 7.4 \times 10^{732}$ and applying Lemma 2.7 to the inequality (5.17) for all $k \in [3, 7100]$, we conclude that $\ell \leq 5000$. This bound for ℓ implies that $n < m < 1.3 \times 10^{47}$ (see Lemma 5.4). The same reasoning as in the previous paragraph applies here to get that $k \in [3, 500]$, and later $\Gamma = \ell \leq 360$. This is impossible.

5.7.2.2 The case $\theta = m$

In this case, from (5.16) we have that $m/\log m < 2.3 \times 10^{12}\ k^4 \log^2 k$. From this and applying Lemma 2.9 with $T := 2.3 \times 10^{12}\ k^4 \log^2 k$, one gets that $n < 1.5 \times 10^{14}\ k^4 \log^3 k$.

Suppose $k > 350$. Then, $n < 1.5 \times 10^{14}\ k^4 \log^3 k < k^{11} < \varphi^{k/2}$. In particular, we have that $n < \varphi^{k/2}$ and $m < 2^{\ell/2}$. Hence, by Lemma 5.5 (a), we obtain that $k < 1.4 \times 10^{13} \log n$. From this and using that $\log n < 11 \log k$, we arrive at $k < 1.6 \times 10^{14} \log k$, which implies $k < 6 \times 10^{15}$. Thus, $m < 1.9 \times 10^{82}$.

Now, we apply again Lemma 2.7 with $M := 1.9 \times 10^{82}$ to the inequality (5.7). In this case, we obtain that $k \leq 830$, and therefore $m < 4.4 \times 10^{28}$. With this new upper bound for m we repeated the process, obtaining that $k \leq 310$, which is a contradiction.

Finally, suppose that $k \in [3, 350]$. Taking $M_k := \lfloor 1.5 \times 10^{14}\ k^4 \log^3 k \rfloor$ and applying Lemma 2.7 to the inequality (5.17) for each $k \in [3, 350]$, we find that $m \leq 100$. Since $k + 2 \leq n < m$, we get $k < 100$. By recalling that we are in the case $\theta = m$, we have $m \leq \ell/2 \leq \ell + 1$ and so $F_m^{(\ell)} = 2^{m-2}$. Consequently, all is reduced to searching for solutions to the equation

$$P_n^{(k)} = 2^{m-2} \quad \text{in the range} \quad 3 \leq k < 100 \quad \text{and} \quad k + 2 \leq n < 100.$$

Using *Mathematica*, we check that the above equation has no solutions. This completes the analysis in the case $k \in [3, 350]$ and hence the proof of Theorem 5.1.

Chapter 6

$k-$Fibonacci numbers close to a power of 2

In this chapter, we find all the members of $F^{(k)}$ which are close to a power of 2. This work continues and extends the previous work of Chern and Cui which investigated the Fibonacci numbers close to a power of 2.

6.1 Introduction

Many of the arithmetic properties of the Fibonacci sequence have recently been studied in generalized Fibonacci sequences. For instance, to cite only a few examples, Fibonacci numbers, and more generally $k-$Fibonacci numbers, which are repdigits, were studied in [37, 90, 93]. It is also known nowadays that 8 is the largest power of 2 in the Fibonacci sequence, a fact that follows from Carmichael's Primitive Divisor theorem [42]. In 2012, Bravo and Luca [35] extended the above result by determining all powers of 2 which are $k-$Fibonacci numbers. Bravo and Gómez [27] later found all powers of 2 as sums of two $k-$Fibonacci numbers. Following this research line, in 2014, Gómez and Luca [62] showed that there are only two kinds of power of two–classes in a $k-$generalized Fibonacci sequence. In this context two or more terms of $k-$Fibonacci sequences are said to be in the same power of two–class if the largest odd factors of the terms are identical. We refer to [38, 63] for results on the largest prime factor of $k-$Fibonacci numbers.

An integer n is said to be close to a positive integer m if it satisfies

$$|n-m|<\sqrt{m}.$$

This closeness notion was first introduced by Chern and Cui [43] in 2014, and motivated them to find all the Fibonacci numbers which are close to a power of 2. The above result was extended by Hasanalizade [68] who found all sums of two Fibonacci numbers which are close to a power of 2. Inspired by these results, Tripathy and Patel [128] generalized the previous works [43, 68] by searching for the sum of three Fibonacci numbers which are close to a power of 2.

In this chapter, we extend the previous work [43] and look for $k-$Fibonacci numbers which are close to a power of 2. More precisely, we study the Diophantine inequality

$$|F_n^{(k)}-2^m|<2^{m/2}, \tag{6.1}$$

in nonnegative integers n,k,m with $k\geq 2$ and $n\geq 1$. If the $k-$Fibonacci number involved in (6.1) equals 1, we then assume that its index is 2 in order to avoid trivial cases. Our main result is the following.

Theorem 6.1. *The Diophantine inequality (6.1) has two parametric families of solutions (n,k,m) with $n,k\geq 2$ and $m\geq 0$, namely*

(a) $(n,k,m)=(t,k,t-2)$ *for* $2\leq t\leq k+1$, *and*

(b) $(n,k,m)=(k+2+t,k,k+t)$ *for* $0\leq t\leq \max\{x\in\mathbb{Z}: 2+x<2^{1+(k-x)/2}\}$.

(c) *In addition, we have the sporadic solution* $(n,k,m)=(12,3,9)$.

We give a brief description of our method. We use Theorem 2.9 to bound n and m polynomially in terms of k. For k small, we apply Lemma 2.7 to lower such bounds to cases that allow us to treat our problem computationally. For large values of k, we apply some ideas developed in [27, 35] for dealing with Diophantine equations involving $k-$Fibonacci numbers.

6.2 Initial considerations

Assume throughout that (n,k,m) is a solution of inequality (6.1). Suppose further that $k\geq 3$ since the case $k=2$ was already studied by Chern and Cui in [43]. A straightforward computation shows that the solutions of (6.1) with $m \leq 3$ belong to the parametric solutions described in Theorem 6.1 (a). So, from now on, we assume that $m \geq 4$.

6.3 Bounds on m in terms of n

Let us now get a relation between n and m. Combining (6.1) with (2.9) and the left inequality of (2.19), one gets that

$$2^{m-1} < F_n^{(k)} \leq 2^{n-2} \qquad \text{and} \qquad \alpha^{n-2} \leq F_n^{(k)} < 2^{m+1}.$$

Thus $m \leq n-2$ and

$$n < 2 + (m+1)\frac{\log 2}{\log \alpha} < 1.14m + 3.14 < 2m \tag{6.2}$$

for all $m \geq 4$, where we used that fact that $(\log 2)/\log \alpha < 1.14$ for all $k \geq 3$. Hence

$$\frac{n}{2} < m \leq n-2. \tag{6.3}$$

6.4 Case $2 \leq n \leq k+1$

In this case, it follows from (6.1), (6.3) and (2.8) that

$$2^{n-2} - 2^m < 2^{m/2}.$$

Note that, if $m \leq n-3$, then $2^{n-3} = 2^{n-2} - 2^{n-3} \leq 2^{n-2} - 2^m < 2^{m/2} \leq 2^{(n-3)/2}$, which is impossible. Thus $m = n-2$, and so the solutions in this case is the parametric family of solutions given in Theorem 6.1 (a).

6.5 Case $n \geq k+2$ and $m = n-2$

Here $m \geq k$, and we must to solve the inequality

$$|F_{m+2}^{(k)} - 2^m| < 2^{m/2}. \tag{6.4}$$

To do this, we distinguish two cases on m, namely $k \leq m \leq 2k$ and $m > 2k$.

6.5.1 The case $k \leq m \leq 2k$

In this case we have that $k+2 \leq m+2 \leq 2k+2$ and so, by (2.10), we get that

$$F_{m+2}^{(k)} = 2^m - (m+2-k)2^{m-k-1}.$$

By substituting this expression into inequality (6.4) we obtain $m+2-k < 2^{k+1-m/2}$. If we write $m = k+i$ for $0 \leq i \leq k$, then the above inequality is transformed into the simpler inequality $2+i < 2^{1+(k-i)/2}$ to be resolved for the integer i with $0 \leq i \leq k$. Defining s as the largest nonnegative integer for which $2+s < 2^{1+(k-s)/2}$, we have that the triples

$$(n,k,m) \in \{(k+2+t, k, k+t) : 0 \leq t \leq s\}$$

are solutions of inequality (6.1). These solutions correspond to the parametric family of solutions given in Theorem 6.1 (b).

6.5.2 The case $m > 2k$

At this point, we present the following Lemma, which shows that inequality (6.4) has no solutions with $m > 2k$.

Lemma 6.1. *Let $k \geq 2$ be an integer and suppose that $r > 2k$. Then*

$$F_{r+2}^{(k)} \leq 2^r - 2^{r/2}.$$

Proof. We shall prove Lemma 6.1 by induction on r. First, we need to prove that the result holds true for $2k+1 \leq r \leq 3k$. Indeed, using the Cooper and Howard' formula in Lemma 2.3, we have that

$$F_{r+2}^{(k)} = 2^r - (r+2-k)\cdot 2^{r-k-1} + \left[\binom{r+2-2k}{2} - 1\right]\cdot 2^{r-2k-2}$$

for all $2k+1 \leq r \leq 3k$. Writing $r = 2k+i$ with $1 \leq i \leq k$, then we must show that

$$2^{k+i/2} \leq (k+2+i)\cdot 2^{k+i-1} - \left[\binom{i+2}{2} - 1\right]\cdot 2^{i-2} \quad \text{for} \quad 1 \leq i \leq k,$$

which is equivalent to

$$\binom{i+2}{2} - 1 \leq 2^{k+1}(k+2+i-2^{1-i/2}) \quad \text{for} \quad 1 \leq i \leq k. \tag{6.5}$$

Since the function $f(x) = x - 2^{1-x/2}$ is increasing, we get that

$$\min\{k + 2 + i - 2^{1-i/2} : 1 \leq i \leq k\} = k + 3 - \sqrt{2}.$$

On the other hand, it is a simple matter to show that

$$\binom{i+2}{2} - 1 \leq \binom{k+2}{2} - 1 = \frac{k(k+1)}{2}.$$

Consequently, to prove (6.5), it is sufficient to show that

$$k(k+1) \leq 2^{k+2}(k + 3 - \sqrt{2}),$$

which clearly holds for all $k \geq 2$. This proves the base case on the induction. Now suppose that $r > 3k$ and that the lemma holds for all i such that $i \leq r - 1$. It then follows from the recurrence relation for $F^{(k)}$ that

$$\begin{aligned} F_{r+2}^{(k)} &= F_{r+1}^{(k)} + F_r^{(k)} + \cdots + F_{r+2-k}^{(k)} \\ &\leq 2^{r-k}(1 + 2 + \cdots + 2^{k-1}) - 2^{r/2}(1/\sqrt{2} + (1/\sqrt{2})^2 + \cdots + (1/\sqrt{2})^k) \\ &\leq 2^r - 2^{r/2}, \end{aligned}$$

as desired. Thus, Lemma 6.1 is proved. □

Finally, note that if (6.4) is satisfied for some $m > 2k$, then we should have that $2^m - 2^{m/2} < F_{m+2}^{(k)}$ which contradicts Lemma 6.1. Thus, inequality (6.4) has no solutions with $m > 2k$.

6.6 Case $n \geq k+2$ and $m \leq n-3$

Using once again (6.1) and (2.16) we get that

$$|f_k(\alpha)\alpha^{n-1} - 2^m| \leq |F_n^{(k)} - f_k(\alpha)\alpha^{n-1}| + |F_n^{(k)} - 2^m| < 2^{m/2} + \frac{1}{2},$$

giving

$$|f_k(\alpha) \cdot \alpha^{n-1} \cdot 2^{-m} - 1| < \frac{2}{2^{m/2}}. \tag{6.6}$$

In order to use the result of Matveev Theorem 2.9, we take $t := 3$ and

$$(\eta_1, b_1) := (f_k(\alpha), 1), \qquad (\eta_2, b_2) := (\alpha, n-1), \qquad (\eta_3, b_3) := (2, -m).$$

We begin by noticing that the three numbers η_1, η_2, η_3 are positive real numbers and belong to $\mathbb{K} = \mathbb{Q}(\alpha)$, so we can take $D := [\mathbb{K} : \mathbb{Q}] = k$. The left–hand side of (6.6) is not zero. Indeed, if this were zero, we would then get that $f_k(\alpha) = 2^m \cdot \alpha^{-(n-1)}$ and so $f_k(\alpha)$ would be an algebraic integer, contradicting (2.14). Note that α^{-1} is an algebraic integer because it is a root of the monic polynomial $z^k\Psi_k(1/z) \in \mathbb{Z}[z]$.

Since $h(f_k(\alpha)) < 2\log k$, $h(\alpha) < (\log 2)/k$ (by (2.29)) and $h(2) = \log 2$, then we can take $A_1 = 2k\log k$, $A_2 = \log 2$ and $A_3 = k\log 2$. Finally, by recalling that $m \leq n-3$, we can take $B := n-1$. Then, Matveev's theorem together with a straightforward calculation gives

$$\left|f_k(\alpha) \cdot \alpha^{n-1} \cdot 2^{-m} - 1\right| > \exp(-5.51 \times 10^{11} k^4 \log^2 k \log n), \tag{6.7}$$

where we used that $1 + \log k \leq 2\log k$ and $1 + \log(n-1) \leq 2\log n$ for all $n \geq k+2$ and $k \geq 3$. Comparing (6.6) and (6.7), taking logarithms and then performing the respective calculations, we get that

$$m < 1.59 \times 10^{12} k^4 \log^2 k \log n.$$

Additionally, by (6.3) we have that $n < 2m$. So that

$$\frac{n}{\log n} < 3.18 \times 10^{12} k^4 \log^2 k. \tag{6.8}$$

We next use the fact that the inequality $x/\log x < A$ implies $x < 2A\log A$ whenever $A \geq 3$ in order to get an upper bound for n depending on k. Indeed, taking $x := n$ and $A := 3.18 \times 10^{12} k^4 \log^2 k$, and performing the respective calculations, inequality (6.8) yields $n < 1.98 \times 10^{14} k^4 \log^3 k$. We record what we have proved so far as a lemma.

Lemma 6.2. *If* (n, k, m) *is a solution of inequality* (6.1) *with* $n \geq k+2$ *and* $m \leq n-3$, *then*

$$m + 3 \leq n < 1.98 \times 10^{14} k^4 \log^3 k.$$

6.6.1 Subcase $k > 170$

In this case, the following inequalities hold

$$n < 1.98 \times 10^{14} k^4 \log^3 k < 2^{k/2}.$$

We now use (6.1) and Lemma 2.4 (applied to $r := n < 2^{k/2}$) to obtain

$$|2^{n-2} - 2^m| = |(2^{n-2} - F_n^{(k)}) + (F_n^{(k)} - 2^m)| < \frac{2^{n-2}}{2^{k/2}} + 2^{m/2},$$

and so

$$|1-2^{m-(n-2)}| < \frac{1}{2^{k/2}} + \frac{1}{2^{n-2-m/2}} < 0.36.$$

In the above we used the fact that $n-2-m/2 \geq 3/2$ and $k > 170$. Since $m \leq n-3$, it follows that $0.5 \leq |1-2^{m-(n-2)}| < 0.36$ which is not possible. Therefore, inequality (6.1) has no solutions for $n \geq k+2$, $m \leq n-3$ and $k > 170$.

6.6.2 Subcase $3 \leq k \leq 170$

In order to apply Lemma 2.7, we put

$$z := (n-1)\log\alpha - m\log 2 + \log f_k(\alpha),$$

and then observe that (6.6) can be written as

$$|e^z - 1| < \frac{2}{2^{m/2}}.$$

Note that $z \neq 0$. If $z > 0$, then we can apply Lemma 2.10 (a) to obtain

$$|z| < |e^z - 1| < 2/2^{m/2}.$$

If, on the contrary, $z < 0$, then $|e^z - 1| < 1/2$ because $m \geq 4$. Thus, by Lemma 2.10 (b) we have that

$$|z| < 2\,|e^z - 1| < 4/2^{m/2}.$$

In any case, we get that $|z| < 4/2^{m/2}$ holds for all $m \geq 4$. Replacing z in the above inequality by its formula and dividing it across by $\log 2$, we get that

$$0 < \left|(n-1)\left(\frac{\log\alpha}{\log 2}\right) - m + \frac{f_k(\alpha)}{\log 2}\right| < 6\cdot(\sqrt{2})^{-m}. \tag{6.9}$$

Putting now

$$\tau := \tau(k) = \frac{\log\alpha}{\log 2}, \quad \mu := \mu(k) = \frac{f_k(\alpha)}{\log 2}, \quad A := 6 \quad \text{and} \quad B := \sqrt{2},$$

the above inequality (6.9) implies

$$0 < |(n-1)\tau - m + \mu| < AB^{-m}. \tag{6.10}$$

It is clear that τ is an irrational number because $\tau > 1$ is a unit in $\mathcal{O}_{\mathbb{K}}$, the ring of integers of $\mathbb{K} = \mathbb{Q}(\alpha)$. So α and 2 are multiplicatively independent. We also put $M_k := \lfloor 1.98 \times 10^{14} k^4 \log^3 k \rfloor$, which is an upper bound on $n-1$ from Lemma 6.2. Applying Lemma 2.7 to the inequality (6.10) for all $k \in [3, 170]$, we obtain that $m < \log(Aq/\varepsilon)/\log B$, where $q = q(k) > 6M_k$ is a denominator of a convergent of the continued fraction of τ such that $\varepsilon = \varepsilon(k) = ||\mu q|| - M_k ||\tau q|| > 0$. A computer search with *Mathematica* revealed that if $k \in [3, 170]$, then the maximum value of $\log(Aq/\varepsilon)/\log B$ is ≤ 660. Hence $m \leq 660$.

6.6.3 The final computation

As we saw in the preceding subsection, it is enough to look for solutions to inequality (6.1) in the range $4 \leq m \leq 660$.

Suppose first that $4 \leq m \leq 60$. Thus, by (6.3), we get that $n < 120$. Here, a brute force search with *Mathematica* in the range

$$7 \leq n < 120, \quad 3 \leq k \leq \min\{170, n-2\} \quad \text{and} \quad 4 \leq m \leq \min\{60, n-3\}$$

gives the sporadic solution $(n, k, m) = (12, 3, 9)$. Now, if $60 < m \leq 660$, then by (6.2) we get that $n < 1.14m + 3.14 < 1.2m$, and so $n < 792$. Then a brute force search done in the range

$$7 \leq n < 792, \quad 3 \leq k \leq \min\{170, n-2\} \quad \text{and} \quad 60 < m \leq \min\{660, n-3\}$$

gives no solutions for the inequality (6.1) with $60 < m \leq 660$. This completes the proof of Theorem 6.1.

Chapter 7

Curious generalized Fibonacci numbers

In this chapter, we find all k–Fibonacci numbers that are curious numbers (i.e., numbers whose base ten representation have the form $a \cdots ab \cdots ba \cdots a$). This work continues and extends a prior result of Trojovský who found all Fibonacci numbers with a prescribed block of digits and a result of Alahmadi et al. who searched for $k-$Fibonacci numbers which are concatenation of two repdigits.

7.1 Introduction

The concept of curious number begins with a problem called "calculator curiosity" which can be found in the recreational mathematics book *Professor Stewart's Hoard of Mathematical Treasures* [126]. Such a problem proposes the reader to check the following equalities:

$$\begin{aligned}
(8 \times 8) + 13 &= 77\\
(8 \times 88) + 13 &= 717\\
(8 \times 888) + 13 &= 7117\\
(8 \times 8888) + 13 &= 71117\\
(8 \times 88888) + 13 &= 711117\\
(8 \times 888888) + 13 &= 7111117\\
(8 \times 8888888) + 13 &= 71111117.
\end{aligned}$$

The numbers on the right side of the equalities above are examples of what we call *curious numbers*. Formally, given a couple of nonnegative integers ℓ and m, we shall define the (ℓ, m)-*curious number* as a natural number with the following base ten representation

$$\underbrace{a\cdots a}_{\ell}\underbrace{b\cdots b}_{m}\underbrace{a\cdots a}_{\ell},$$

where a and b are integers such that $a, b \in \{0, 1, \ldots, 9\}$. A nonnegative integer is called a curious number if it is an (ℓ, m)-curious number for some integers ℓ, m with $\ell \geq 0$ and $m \geq 1$. Note that a $(0, m)-$curious number is not more than a repdigit, i.e., a positive integer with only one distinct digit in its decimal representation. The smallest curious number that is not a repdigit is 101. The first curious numbers are

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 88, 99, 101, 111, 121, 131, 141, \ldots,$$

and this matches with the sequence A335779 in Sloane's On-Line Encyclopedia of Integer sequences [123]. Few properties of curious numbers are currently known. For instance, Borade and Mayle [16] determined all curious number that are perfect squares.

Diophantine problems involving repdigits and terms of certain linear recurrence sequences have been recently an active research field in number theory. It should be mentioned that Luca [90] in 2000 showed that 55 and 11 are the largest repdigits in the Fibonacci and Lucas sequences, respectively. Further, Faye and Luca [58] looked for repdigits in the usual Pell sequence and using some elementary methods they concluded that there are no Pell numbers larger than 10 which are repdigits. The above results has been generalized and extended in various directions. For example, a conjecture (proposed by Marques [93]) about repdigits in $k-$Fibonacci sequences was proved by Bravo–Luca [37]. Alahmadi et al. [2] generalized recently the results mentioned above by showing that only repdigits with at least two digits as product of ℓ consecutive $k-$Fibonacci numbers occur only for $(k, \ell) = (2, 1), (3, 1)$, extending the works [19, 94] which dealt with the particular cases of Fibonacci and Tribonacci numbers. Alahmadi et al. [3] determined all $k-$Fibonacci numbers that are concatenations of two repdigits, while Trojovský [131] found all Fibonacci numbers with a prescribed block of digits. Finally, Bravo et al. [17] studied a problem similar to the one worked in [3] but focused on the $k-$Lucas sequence.

Diophantine equations involving sums and products have been also discussed. For example, Erduvan and Keskin [54] found all repdigits expressible as products of two Fibonacci or Lucas numbers. We also mention the work of Normenyo, Luca and Togbé [107] who found all repdigits expressible as sums of three Pell numbers. Shortly afterwards, they extended their work to four Pell numbers [108]. For linear recurrence sequence of order k, it is known that Bravo–Luca [40] found all repdigits which are sums of two $k-$Fibonacci numbers (see [46] for a product version), while Rayaguru and Bravo

[113] determine all repdigits expressible as sums of two $k-$Lucas numbers. The last work generalizes a prior result of Şiar and Keskin [47] who dealt with the above problem for the particular case of Lucas numbers and a result of Bravo and Luca [39] who searched for repdigits that are $k-$Lucas numbers.

In this chapter, we determine all curious numbers which are $k-$Fibonacci numbers, i.e.,

$$F_n^{(k)} = \underbrace{a\cdots a}_{\ell}\underbrace{b\cdots b}_{m}\underbrace{a\cdots a}_{\ell},$$

which continues and extends the works in [3] and [131]. Since curious numbers can be expressed algebraically as

$$\underbrace{a\cdots a}_{\ell}\underbrace{b\cdots b}_{m}\underbrace{a\cdots a}_{\ell} = 10^{\ell+m}a\left(\frac{10^{\ell}-1}{9}\right) + 10^{\ell}b\left(\frac{10^{m}-1}{9}\right) + a\left(\frac{10^{\ell}-1}{9}\right),$$

we look for all the solutions of the Diophantine equation

$$F_n^{(k)} = \frac{1}{9}\left(a\cdot 10^{2\ell+m} - (a-b)\cdot 10^{\ell+m} + (a-b)\cdot 10^{\ell} - a\right) \tag{7.1}$$

in positive integers n, k, m, ℓ, a and b with $k \geq 2$, $a, b \in \{0, 1, \ldots, 9\}$ and $a \neq b$.

Before presenting our main theorem, it is important to mention that in equation (7.1) we assumed $\ell, m \geq 1$ and $a \neq b$ since otherwise the problem reduces to finding all $k-$Fibonacci numbers that are repdigits or concatenations of two repdigits, and these problems have been already solved in [37] and [3], respectively. In addition, note that when $a = 0$, our problem also reduces to determining all $k-$Fibonacci numbers that are concatenations of two repdigits. Thus, throughout this paper we also assume that $a \geq 1$. Our result is the following.

Theorem 7.1. *The only curious generalized Fibonacci number is* $F_{11}^{(5)} = 464$.

As an immediate consequence of Theorem 7.1 we have the following corollary.

Corollary 7.1. *There are no curious numbers that are powers of* 2.

7.2 Initial considerations

Assume throughout that (n, k, a, b, ℓ, m) is a solution of the equation (7.1). First, we note that $n \leq 3$ is impossible since $F_n^{(k)}$ must have at least 3 digits in its decimal representation. Thus, we assume $n \geq 4$. We now want to establish a relationship between the variables of (7.1). For this purpose, we combine inequalities (2.9) and (2.19) in equation (7.1) to get

$$10^{2\ell+m-1} < F_n^{(k)} \leq 2^{n-2} \qquad \text{and} \qquad \alpha^{n-2} \leq F_n^{(k)} < 10^{2\ell+m}, \tag{7.2}$$

from which it follows that

$$2\ell + m < (n-2)\left(\frac{\log 2}{\log 10}\right) + 1 \quad \text{and} \quad n - 2 < (2\ell + m)\left(\frac{\log 10}{\log \alpha}\right).$$

In particular

$$(2\ell + m) + 2 < n < 6(2\ell + m) \quad \text{holds for all} \quad n \geq 4. \tag{7.3}$$

7.3 Powers of two which are curious numbers

Assuming $4 \leq n \leq k+1$ and taking into account (2.8), we can rewrite equation (7.1) as

$$a \cdot 10^{2\ell+m} - (a-b) \cdot 10^{\ell+m} + (a-b) \cdot 10^{\ell} - 9 \cdot 2^{n-2} = a. \tag{7.4}$$

Since $\ell < n-2$ by (7.3), it follows from (7.4) that $2^\ell \mid a$ and so $\ell \leq 3$. We now use (7.4) once again to obtain that

$$a \cdot 10^{2\ell+m} - (a-b) \cdot 10^{\ell+m} - 9 \cdot 2^{n-2} = a - (a-b) \cdot 10^{\ell} \in \mathcal{R}, \tag{7.5}$$

where $\mathcal{R} := ([-8991, 0) \cup (0, 8001]) \cap \mathbb{Z}$.

Now, since the largest 2−adic valuation[1] of integers of the interval $\mathcal{R}$ is 7, we get that $\ell + m \leq 7$ by (7.5). So, $m \leq 6$. Finally, a numerical check with *Mathematica* revealed that equation (7.1) has no solutions in the range

$$4 \leq n \leq k+1, \quad 1 \leq \ell \leq 3, \quad 1 \leq m \leq 6, \quad 1 \leq a \leq 9 \quad \text{and} \quad 0 \leq b \leq 9.$$

Thus, from now on we suppose that $n \geq k+2$.

[1]The 2−*adic valuation* of a positive integer number n is the exponent of the greatest power of 2 that divides n.

7.4 Bounding n in terms of k

In this subsection we want to find an upper bound for n in terms of k. To do this, we put $X \coloneqq a \cdot 10^{\ell} - (a-b)$ and rewrite (7.1) using (2.17) in two different forms, namely

$$\begin{aligned} 9f_k(\alpha)\alpha^{n-1} - a \cdot 10^{2\ell+m} &= -9e_k(n) - (a-b)\cdot 10^{\ell+m} + (a-b)\cdot 10^{\ell} - a, \qquad \text{and} \\ 9f_k(\alpha)\alpha^{n-1} - 10^{\ell+m}X &= -9e_k(n) + (a-b)\cdot 10^{\ell} - a. \end{aligned} \tag{7.6}$$

For future calculations it will be important to note that

$$1 \le X \le 10^{\ell+1}. \tag{7.7}$$

We now take absolute value in relations given by (7.6) and doing some straightforward calculations we obtain

$$\begin{aligned} \left|9f_k(\alpha)\alpha^{n-1} - a\cdot 10^{2\ell+m}\right| &< 11\cdot 10^{\ell+m}, \qquad \text{and} \\ \left|9f_k(\alpha)\alpha^{n-1} - 10^{\ell+m}X\right| &< 11\cdot 10^{\ell}. \end{aligned} \tag{7.8}$$

Dividing both sides of each one of the above inequalities (7.8) by $a \cdot 10^{2\ell+m}$ and $10^{\ell+m}X$, respectively, and rearranging some terms, we get

$$\left|\alpha^{n-1}\cdot 10^{-(2\ell+m)}\cdot\left(\tfrac{9f_k(\alpha)}{a}\right) - 1\right| < 11/10^{\ell}, \qquad \text{and} \tag{7.9}$$

$$\left|\alpha^{n-1}\cdot 10^{-(\ell+m)}\cdot\left(\tfrac{9f_k(\alpha)}{X}\right) - 1\right| < 11/10^{m}. \tag{7.10}$$

At this point, we claim that the left–hand side of (7.9) and (7.10) are not zero. Indeed, if these were zero, we would obtain respectively

$$a\cdot 10^{2\ell+m} = 9f_k(\alpha)\alpha^{n-1} \quad \text{and} \quad 10^{\ell+m}X = 9f_k(\alpha)\alpha^{n-1}.$$

Conjugating the above equalities with an automorphism σ of the Galois group of $\Psi_k(x)$ over $\mathbb{Q}$ such that $\sigma(\alpha) = \alpha_i$ for some $i > 1$, taking absolute values and using the fact that $|9f_k(\alpha_i)\alpha_i^{n-1}| < 9$, we obtain

$$a\cdot 10^{2\ell+m} < 9 \quad \text{and} \quad 10^{\ell+m}X < 9,$$

respectively. However, these lead to a contradiction since

$$a\cdot 10^{2\ell+m} \ge 10^3 \quad \text{and} \quad 10^{\ell+m}X \ge 10^2.$$

We shall now apply Matveev's Theorem on inequalities (7.9) and (7.10) (in that order). To do this, we take the following parameters:

$$t := 3, \quad \eta_1 := \alpha, \quad \eta_2 := 10, \quad \eta_{3,1} := 9f_k(\alpha)/a, \quad \eta_{3,2} := 9f_k(\alpha)/X,$$
$$b_1 := n-1, \quad b_{2,1} := -(2\ell+m), \quad b_{2,2} := -(\ell+m), \quad b_3 := 1.$$

The real number field containing $\eta_1, \eta_2, \eta_{3,1}, \eta_{3,2}$ is $\mathbb{K} := \mathbb{Q}(\alpha)$. From this and (7.3), we can take $D := [\mathbb{K} : \mathbb{Q}] = k$ and $B := n$ in any application of Matveev's Theorem.

On the other hand, since $h(\eta_1) < (\log 2)/k$ (by (2.29)) and $h(\eta_2) = \log 10$, we can always take $A_1 := \log 2$ and $A_2 := k \log 10$. Furthermore, from Lemma 2.11 and (7.7) we get that

$$h(\eta_{3,1}) < 6 \log k, \qquad \text{and} \tag{7.11}$$
$$h(\eta_{3,2}) < 2\log k + (\ell+1)\log 10. \tag{7.12}$$

7.4.1 An inequality for ℓ in terms of k

In order to apply Matveev's Theorem on (7.9) with the parameters η_1, η_2 and $\eta_{3,1}$, we take A_1, A_2 as mentioned before and $A_3 := 6k \log k$ (by (7.11)) to obtain

$$\left| \alpha^{n-1} \cdot 10^{-(2\ell+m)} \cdot \left(\frac{9f_k(\alpha)}{a} \right) - 1 \right| > \exp\left(-9 \times 10^{12} k^4 \log^2 k \log n\right), \tag{7.13}$$

where we used that $1+\log k < 3\log k$ and $1+\log n < 2\log n$ hold for all $k \geq 2$ and $n \geq 4$, respectively. Comparing (7.9) and (7.13) and performing the respective calculations, we get

$$\ell < 4 \times 10^{12}\, k^4 \log^2 k \log n. \tag{7.14}$$

7.4.2 An inequality for m in terms of k

In the light of (7.12) and (7.14) we deduce that

$$h(\eta_{3,2}) < 10^{13}\, k^4 \log^2 k \log n.$$

This allows us to choose now $A_3 := 10^{13}\, k^5 \log^2 k \log n$. We then apply Matveev's Theorem on (7.10) with the parameters η_1, η_2 and $\eta_{3,2}$ to get

$$\left| \alpha^{n-1} \cdot 10^{-(\ell+m)} \cdot \left(\frac{9f_k(\alpha)}{X} \right) - 1 \right| > \exp\left(-2 \times 10^{25}\, k^8 \log^3 k \log^2 n\right). \tag{7.15}$$

Using now (7.10) and (7.15), it follows that

$$m < 10^{25}\, k^8 \log^3 k \log^2 n. \tag{7.16}$$

7.4.3 An inequality for n in terms of k

We finally use (7.14) and (7.16) combined with (7.3) to assert that

$$\frac{n}{\log^2 n} < 1.2 \times 10^{26}\, k^8 \log^3 k. \tag{7.17}$$

In order to find an upper bound on n in terms of k and $\log k$, we apply Lemma 2.9 with parameters $T := 1.2 \times 10^{26}\, k^8 \log^3 k$ and $m := 2$. From the above, we get from (7.17) the following lemma.

Lemma 7.1. *If (n, k, a, b, ℓ, m) is a solution of equation (7.1) with $n \geq k+2$, then*

$$2\ell + m < n < 5 \times 10^{30}\, k^8 \log^5 k.$$

7.5 The case of large k

Suppose that $k > 430$. Note that for such values of k we have

$$5 \times 10^{30}\, k^8 \log^5 k < 2^{k/2}.$$

Then by Lemma 7.1, we get that the inequality $n < 2^{k/2}$ is satisfied when $k > 430$ and therefore we are in the hypothesis of Lemma 2.4. Applying the above lemma and equation (7.1) we obtain

$$\left|\frac{a}{9} \cdot 10^{2\ell+m} \cdot 2^{-(n-2)} - 1\right| < \frac{3 \cdot 10^{\ell+m}}{2^{n-2}} + \frac{1}{2^{k/2}} < \frac{30}{10^{\ell}} + \frac{1}{2^{k/2}},$$

where we have used that $10^{\ell+m}/2^{n-2} < 10/10^{\ell}$ (see (7.2)). Consequently

$$\left|\frac{a}{9} \cdot 10^{2\ell+m} \cdot 2^{-(n-2)} - 1\right| < \frac{30}{2^{\ell\theta}} + \frac{1}{2^{k/2}} \leq \frac{31}{2^{\lambda}}, \tag{7.18}$$

where $\theta := (\log 10)/(\log 2)$ and $\lambda := \min\{k/2, \ell\theta\}$. Again, in order to use the result of Matveev, we take $t := 3$,

$$(\eta_1, b_1) := (a/9, 1), \quad (\eta_2, b_2) := (10, 2\ell + m) \quad \text{and} \quad (\eta_3, b_3) := (2, -(n-2)).$$

We begin by noticing that the three numbers η_1, η_2, η_3 are positive rational numbers, so we can take $\mathbb{K} := \mathbb{Q}$ for which $D := 1$. To see why the left–hand side of (7.18) is not zero, note that otherwise, we would get that $a \cdot 10^{2\ell+m} = 9 \cdot 2^{n-2}$ which is impossible since its left-hand side is divisible by 5 while its right-hand side is not.

Clearly, we can take $A_1 := \log 9$, $A_2 := \log 10$ and $A_3 := \log 2$. Here, we can also take $B := n$. Then, Matveev's Theorem together with a straightforward calculation gives

$$\left|\frac{a}{9} \cdot 10^{2\ell+m} \cdot 2^{-(n-2)} - 1\right| > \exp\left(-1.1 \times 10^{12} \log n\right), \tag{7.19}$$

where we used again that $1 + \log n < 2 \log n$ holds for all $n \geq 4$. Comparing (7.18) and (7.19), taking logarithms and then performing the respective calculations, we arrive at

$$\lambda < 1.8 \times 10^{12} \log n.$$

Note that, if $\lambda = k/2$, then $k < 3.6 \times 10^{12} \log n$. Since $\log n < 73 \log k$ holds for all $k > 430$ by Lemma 7.1, we get $k < 2.7 \times 10^{14} \log k$ giving $k < 10^{16}$. For the case when $\lambda = \ell\theta$, we have $\ell < 5.5 \times 10^{11} \log n$. Here, proceeding as in (7.18), we obtain

$$\left|\frac{X}{9} \cdot 10^{\ell+m} \cdot 2^{-(n-2)} - 1\right| < \frac{2^{\ell\theta}}{2^{n-2}} + \frac{2}{2^{k/2}} \leq \frac{2^{k/2}}{2^k} + \frac{2}{2^{k/2}} = \frac{3}{2^{k/2}}. \tag{7.20}$$

The same argument used before also shows that the left-hand side of (7.20) is not zero. With a view towards applying Matveev's Theorem, we take the same parameters as in the previous application, except by η_1 and b_2 which in this case are given by $X/9$ and $\ell + m$, respectively. As before, $\mathbb{K} := \mathbb{Q}$, $D := 1$, $A_2 := \log 10$, $A_3 := \log 2$ and $B := n$. Moreover, by (7.7), we have that

$$h(\eta_1) = \log X \leq (\ell + 1) \log 10 < 1.3 \times 10^{12} \log n.$$

Hence, we can take $A_1 := 1.3 \times 10^{12} \log n$. This time Matveev's theorem leads to

$$\exp\left(-6 \times 10^{23} \log^2 n\right) < \left|\frac{X}{9} \cdot 10^{\ell+m} \cdot 2^{-(n-2)} - 1\right| < \frac{3}{2^{k/2}},$$

which implies $k < 9.4 \times 10^{26} \log^2 k$. Hence $k < 5 \times 10^{30}$, and so by Lemma 7.1 we get that $n < 3.5 \times 10^{285}$. At this point, we shall summarize what we have obtained so far on the upper bounds for k and n. The result is the following.

Lemma 7.2. *If (n, k, a, b, ℓ, m) is a solution of equation (7.1) with $k > 430$ and $n \geq k+2$, then inequalities*

$$k < 5 \times 10^{30} \quad \text{and} \quad 2\ell + m < n < 3.5 \times 10^{285}$$

hold.

7.6 Reducing the bound on k

We now want to reduce our bound on k by using Lemma 2.7. Let

$$\Gamma_1 := \log(a/9) + (2\ell + m)\log 10 - (n-2)\log 2.$$

Then, from (7.18) we get that $|e^{\Gamma_1} - 1| < 31/2^{\lambda}$. Note that $31/2^{\lambda} < 1/2$ whenever $\lambda \geq 6$. Now, assuming that $\lambda \geq 6$, we obtain $|e^{\Gamma_1} - 1| < 1/2$ and so Lemma 2.10 shows that $0 < |\Gamma_1| < 2|e^{\Gamma_1} - 1| < 62/2^{\lambda}$. Dividing the above inequality through $\log 2$ gives

$$0 < |(2\ell + m)\theta - n + \mu_a| < 90 \cdot 2^{-\lambda} \qquad \text{for all} \qquad \lambda \geq 6, \tag{7.21}$$

where $\mu_a := 2 + (\log(a/9))/(\log 2)$. Taking $M := 3.5 \times 10^{285}$ we get that $2\ell + m < M$. Applying now Lemma 2.7 to inequality (7.21) for each $a \in \{1, 2, \ldots, 8\}$, we found with the help of *Mathematica* that $\lambda \leq 960$.

For the case $a = 9$, we can not use Lemma 2.7 because the corresponding value of ϵ is always negative. However, one can see that if $a = 9$, then the resulting inequality from (7.21) has the shape

$$|x\gamma - y| < 90 \cdot 2^{-\lambda}, \tag{7.22}$$

with $\gamma := \theta$ being an irrational number and $x := 2\ell + m$, $y := n - 2 \in \mathbb{Z}$. In order to apply Lemma 2.8 on the left-hand side of (7.22), we define $[a_0, a_1, a_2, a_3, \ldots] = [3, 3, 9, 2, \ldots]$ as the continued fraction of γ and p_i/q_i its ith convergent. We can also take $M := 3.5 \times 10^{285}$ so that $x < M$ by Lemma 7.2. A quick inspection using *Mathematica* reveals that $q_{573} \leq M < q_{574}$ and therefore $a_M := \max\{a_i \mid 0 \leq i \leq 574\} = a_{135} = 5393$. Hence, by Lemma 2.8, we obtain that $|x\gamma - y| > 1/(5395(2\ell + m))$ and after a comparison with (7.22), we get that $\lambda \leq 967$. Thus, $\lambda \leq 967$ always holds.

Note that if $\lambda = k/2$, then $k \leq 1934$. On the other hand, if $\lambda = \ell\theta$ then we have $\ell \leq 291$. Now, we put

$$\Gamma_2 := (\ell + m)\log 10 - (n-2)\log 2 + \log(X/9).$$

Here, (7.20) yields $|e^{\Gamma_2} - 1| < 3/2^{k/2}$. Since $k > 430$, we get that $|e^{\Gamma_2} - 1| < 1/2$. Using Lemma 2.10 again, we deduce that $0 < |\Gamma_2| < 6/2^{k/2}$. Dividing through the above inequality by $\log 2$ gives

$$0 < |(\ell + m)\theta - n + \mu(a, b, \ell)| < 9 \cdot 2^{-k/2}, \tag{7.23}$$

where $\mu(a, b, \ell) := 2 + (\log(X/9)/(\log 2))$. Here, we also take $M := 3.5 \times 10^{285}$ and apply Lemma 2.7 to inequality (7.23) for all $a, b \in \{0, 1, \ldots, 9\}$ with $a \geq 1$, $a \neq b$ and $1 \leq \ell \leq 291$, except when

$$(a, b, \ell) \in \{(1,0,1), (1,9,1), (2,0,1), (3,9,1), (4,0,1), (7,9,1), (8,0,1), (4,9,1), (5,0,1)\}$$

and $(a,b,\ell)=(9,9,\ell)$ for all $\ell\geq 1$. Indeed, A computer search with *Mathematica* revealed that $k\leq 1955$. Now, we deal with the special cases mentioned just before. First of all, it is a straightforward exercise to check that in these cases we have

$$\mu(a,b,\ell)=\begin{cases} 2, & \text{if } (a,b,\ell)=(1,0,1);\\ 3, & \text{if } (a,b,\ell)=(1,9,1),(2,0,1);\\ 4, & \text{if } (a,b,\ell)=(3,9,1),(4,0,1);\\ 5, & \text{if } (a,b,\ell)=(7,9,1),(8,0,1);\\ 1+\theta, & \text{if } (a,b,\ell)=(4,9,1),(5,0,1);\\ 1+\ell\theta, & \text{if } (a,b,\ell)=(9,9,\ell),\ \ell\geq 1.\end{cases}$$

In such cases, the inequality (7.23) turns into

$$|(m+1)\theta-(n-i)|<9\cdot 2^{-k/2}\qquad (\text{for}\quad i=2,3,4,5),\quad \text{or}$$
$$|(m+2)\theta-(n-1)|<9\cdot 2^{-k/2},\qquad \text{or}\qquad |(2\ell+m)\theta-(n-1)|<9\cdot 2^{-k/2}.$$

In any case, by the same arguments used to get inequality (7.22), we obtain $2^{k/2}<1.7\times 10^{290}$, which implies that $k\leq 1928$. Thus, $k\leq 1955$ holds for any choice of λ. Then, by Lemma 7.1, $2\ell+m<2.7\times 10^{61}\coloneqq M$. With this new choice of M, Lemma 2.7 applied to inequality (7.21) implies that $\lambda\leq 222$ (including the case $a=9$). If $\lambda=k/2$, then $k\leq 444$, while if $\lambda=\ell\theta$, we have that $\ell\leq 66$. We finally apply Lemma 2.7 with $M\coloneqq 2.7\times 10^{61}$ to inequality (7.23) for all $a,b\in\{0,1,\ldots,9\}$ with $a\neq b$, $a\geq 1$ and $1\leq\ell\leq 66$, except in the special cases mentioned above. With the help of *Mathematica* we found that $k\leq 457$. The same upper bound for k holds in the special cases. So, $k\leq 457$ holds for any choice of λ. Finally, taking $M\coloneqq 8.2\times 10^{55}$ and repeating the previous procedure, we obtain that $k\leq 422$, which is a contradiction. Hence, the equation (7.1) has no solutions for $k>430$.

7.7 The case of small k

Suppose now that $k\in[2,430]$. Note that for each of these values of k, Lemma 7.1 gives us absolute upper bounds for n. However, these upper bounds are so large and will be reduced by using Lemma 2.7 once again. To do this we put

$$\Omega\coloneqq\begin{bmatrix}\Omega_1\\ \Omega_2\end{bmatrix}=\begin{bmatrix}(n-1)\log\alpha-(2\ell+m)\log 10+\log((9f_k(\alpha))/a)\\ (n-1)\log\alpha-(\ell+m)\log 10+\log((9f_k(\alpha))/X)\end{bmatrix}.$$

Thus, (7.9) and (7.10) can be rewritten respectively as

$$|e^{\Omega_1}-1|<\frac{11}{10^{\ell}},\qquad\text{and}\qquad |e^{\Omega_2}-1|<\frac{11}{10^{m}}.$$

Now assuming $\ell \geq 2$ and $m \geq 2$, we see that $|e^{\Omega_i} - 1| < 1/2$ for all $i \in \{1, 2\}$. Using Lemma 2.10, we deduce that

$$0 < |\Omega_1| < 22/10^{\ell} \qquad \text{and} \qquad 0 < |\Omega_2| < 22/10^{m}.$$

Dividing both inequalities by $\log 10$, we get

$$0 < |(n-1)\tau_1 - (2\ell + m) + \mu_1(k, a)| < 10 \cdot 10^{-\ell}, \qquad \text{and} \tag{7.24}$$
$$0 < |(n-1)\tau_1 - (\ell + m) + \mu_2(k, \ell, a, b)| < 10 \cdot 10^{-m}, \tag{7.25}$$

where

$$\tau_1 := \frac{\log \alpha}{\log 10} \qquad \text{and} \qquad \begin{bmatrix} \mu_1(k, a) \\ \mu_2(k, \ell, a, b) \end{bmatrix} := \begin{bmatrix} \frac{\log((9 f_k(\alpha))/a)}{\log 10} \\ \frac{\log((9 f_k(\alpha))/X)}{\log 10} \end{bmatrix}.$$

Note that τ_1 clearly is an irrational number because α and 10 are multiplicatively independent. Next, we shall apply Lemma 2.7 to (7.24) and (7.25). For this purpose, we put also $M_k := 5 \times 10^{30}\, k^8 \log^5 k$, which is an upper bound on $n - 1$ by Lemma 7.1.

In the first application, we choose the following parameters

$$\tau := \tau_1, \qquad \mu := \mu_1(k, a), \qquad A := 10 \qquad \text{and} \qquad B := 10.$$

A computer search with *Mathematica* revealed that if $k \in [2, 430]$ and $a \in \{1, 2, \ldots, 9\}$, then the maximum value of $\lfloor \log(Aq/\epsilon)/\log B \rfloor$ is 130. Then, every possible solution (n, k, a, b, ℓ, m) of equation (7.1) for which $(k, a) \in [2, 430] \times [1, 9]$ has $\ell \in [1, 130]$.

For the second application, we take

$$\tau := \tau_1, \qquad \mu := \mu_2(k, \ell, a, b), \qquad A := 10 \qquad \text{and} \qquad B := 10.$$

In this case, *Mathematica* shows that for each $a, b \in \{0, 1, \ldots, 9\}$, $a \geq 1$, $a \neq b$, $k \in [2, 430]$ and $\ell \in [1, 130]$, the maximum value of $\lfloor \log(Aq/\epsilon)/\log B \rfloor$ is 130. Thus,

$$m \in [1, 130] \quad \text{and so} \quad n \in [1, 2340].$$

Finally, we use *Mathematica* to display the values of $F_n^{(k)}$ for $(k, n) \in [2, 430] \times [4, 2340]$, and check that the equation (7.1) has only the solution listed in Theorem 7.1. This completes the analysis in the case $k \in [2, 430]$ and ends the proof.

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